Spectral Sequences and Group Cohomology

Alejandro De Las Peñas Castaño

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Introduction

In essence, spectral sequences are computational tools used to compute (co)homology groups by taking successive approximations. They contain a lot of information in such a compact definition that at first, spectral sequences may seem too complicated to be of any use, but this could not be farther from the truth.

Spectral sequences were first introduced by Jean Leray in 1946 while captive in Austria during World War II. His mathematical upbringing was in analysis and differential equations (his phd thesis was on the Navier-Stokes equation), but switched to "pure mathematics" to avoid being enlisted by the germans to help in their war machine. However, Leray did not stray too far from his origins. His work on spectral sequences was motivated by trying to generalize/axiomatize Cartan's and de Rham's theory of differential forms to topological spaces. In a series of two terse notes in 1946, Leray introduced the definition of sheaves and spectral sequences. These papers are somewhat opaque and the modern definitions of these objects are quite different.

The modern notion of sheaf is due to Cartan who was himself motivated by André Weil's modern proof of de Rham's theorem. Though Weil himself does not mention Leray when discussing the inspiration for this proof, in a 1947 letter to Cartan, it is very likely that he discussed this problem personally with Leray in the previous year.

The modern notion of spectral sequence is due to Koszul in 1947-1948, a student of Cartan, who "liberated the notion of spectral sequence from its topological confinement, and brought the thoery to its present-day form" [Mil00]. Most of the modern notation used in the theory of spectral sequences is due to Koszul.

It is not clear why the word "spectral" was chosen by Leray. During the late 40's, people had been referring to spectral sequences as the Leray-Koszul sequence, but, according to Borel, Leray wanted terminology without proper names and "sequence of graded differential algebras" was too long. Borel then speculates that, since Leray's definition of filtration allowed them to be parametrized by real numbers, spectral sequences, to Leray, felt formally reminiscent of "things labeled spectral" in Analysis.

The theory of spectral sequences in the late 40's wasn't very well accepted by the mathematical community since Leray's original notes were hard to understand and included no applications or theorems. This changed however, when in 1950 a paper of Borel and Serre applied Leray's theory to show that Euclidean space cannot admit a proper fibration by connected fibers over a compact base space. More precisely, there does not exist a continuous map $p : \mathbb{R}^n \to B$, where B is compact, $p^{-1}(b) \subset \mathbb{R}^n$ is connected and such that p is a fibration, i.e. it satisfies the homotopy lifting property: for every topological space X and every homotopy $h : X \times [0,1] \to B$ there exists a lift $\widehat{h} : X \times [0,1] \to \mathbb{R}^n$ that preserve any choice of map $h_0 : X \to \mathbb{R}^n$ and makes the following diagram commute:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{h_0} & \mathbb{R}^n \\ \big\downarrow & \overset{\widehat{h}}{\nearrow} & \big\downarrow p \\ X \times [0,1] & \xrightarrow{h} & B \end{array}$$

If you restrict the above definition to X's that are CW-complexes, then p is called a Serre fibration.

The final result result that turned the theory of spectral sequences into its modern form is Serre's PhD thesis where, among other things, he constructs a spectral sequence that computes the singular (co)homology of the total space X of a Serre fibration $p : X \to B$ in terms of the (co)homology of B and the fiber F. More precisely, he showed that

$$H^p_{\text{sing}}(B, H^q_{\text{sing}}(F)) \Longrightarrow H^{p+q}(X).$$

See definition 9 for a precise statement about the convergence of spectral sequences. The above convergence is true for general Serre fibrations with some slight modifications; we've written a simplified version when B is simply connected so the choice of fiber doesn't matter since all fibers are homotopy equivalent.

Soon, many different examples of spectral sequences were found. Below we informally list a couple of them that are important in algebraic geometry.

1. (Serre-Hochschild) Let G be a group, $N \trianglelefteq G$ and A a G-module, then

$$H^p(G/N, H^q(N, A)) \Longrightarrow H^{p+q}(G, A)$$

We will study this case in detail in Section 4.

2. (Čech-to-derived-functor) Let $\mathcal{F}$ be a presheaf on a topological space X with a fixed open cover $\mathcal{U} = \{U_i\}_{i \in I}$. Then we have the *Čech complex*

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{i_0, \dots, i_q} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q})$$

which can be made into a cochain complex $C^\bullet(\mathcal{U}, \mathcal{F})$. Then the Čech cohomology of $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is defined as

$$\check{H}^q(\mathcal{U}, \mathcal{F}) := H^q(C^\bullet(\mathcal{U}, \mathcal{F})).$$

Then there is a spectral sequence that satisfies

$$\check{H}^p(\mathcal{U}, \mathcal{H}^q(X, \mathcal{F})) \Longrightarrow H^{p+q}(X, \mathcal{F})$$

where $\mathcal{H}^q(X, \mathcal{F})$ is the presheaf $U \mapsto H^q(U, \mathcal{F}|_U)$ and $H^n(X, \mathcal{F})$ is the right derived functor of the global sections functor.

3. (Grothendieck) Let $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be abelian categories such that both $\mathcal{B}$ and $\mathcal{C}$ have enough injectives. Suppose we have two left exact functors

$$\mathcal{B} \xrightarrow{\mathcal{G}} \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D}$$

such that $\mathcal{G}$ sends injective objects I in $\mathcal{B}$ to $\mathcal{F}$-acyclic objects, i.e. $(L_i\mathcal{F})(\mathcal{G}(I)) = 0$ for all $i > 0$, where $\{L_i\mathcal{F}\}_{i \geq 0}$ are the left derived functors of $\mathcal{F}$. Then for every object $B \in \mathcal{B}$ there is a convergent spectral sequence

$$(R^p\mathcal{F})(R^q\mathcal{G}(B)) \Longrightarrow R^{p+q}(\mathcal{F} \circ \mathcal{G})(B).$$

In fact, the Grothendieck spectral sequence generalizes quite a few other spectral sequences like the Hochschild-Serre spectral sequence.

In section 1 we go over the preliminary definitions of filtered abelian groups to set up the objects that will produce for us the spectral sequences we will be interested in. In section 2, we review the basic definitions of spectral sequence and their convergence. We also prove the existence of the exact sequence of edge terms arising from a convergent spectral sequence, which is of great computational value. In section 3 we prove the existence of spectral sequences attached to filtered abelian groups. This section can be mostly skipped on a first reading since it is quite technical without any enlightening ideas or concepts. In section 4 we discuss the Hochschild-Serre spectral sequence in detail and deduce some of its properties which are of importance in Class Field Theory. Finally, in section 5 we discuss an interesting application of spectral sequences to the computation of Euler characteristics of groups.

1 Filtered Abelian Groups

We begin with a description of the main way spectral sequences arise in the theory of Galois cohomology. It starts with filtrations.

Definition 1. A *filtration* $\{F^pA\}_{p \in \mathbb{Z}}$ on an abelian group A is a descending chain of subgroups of A:

$$\dots \supseteq F^{p-1}A \supseteq F^pA \supseteq F^{p+1}A \supseteq \cdots.$$

An abelian group A with a filtration is called a *filtered abelian group*. Furthermore, if A is graded, say $A = \oplus A_n$, then we say that:

- the filtration is *compatible with the grading* if

$$F^pA = \bigoplus_{q \in \mathbb{Z}} (F^pA \cap A_{p+q}).$$

- the filtration is *regular* if for every $n \in \mathbb{Z}$, there exists an integer $\mu = \mu(n) \in \mathbb{Z}$ such that

$$p \geq \mu \quad \Longrightarrow \quad F^pA \cap A_n = 0$$

Remark. If A is a graded abelian group with a compatible filtration $\{F^pA\}_{p\in\mathbb{Z}}$, then each term of the filtration F^pA is itself a graded abelian group with homogeneous components

$$(F^pA)_q := F^pA \cap A_{p+q}.$$

These components hold a lot of information about the original group in the way they interact with both the grading of A and its filtration. More precisely, $(F^pA)_q$ is a subgroup of both the $(p+q)$th homogeneous component of A and the pth part of the filtration which can be recovered by summing $(F^pA)_q$ over all $q \geq 0$.

Example. Graded abelian groups possess a "trivial" filtration that is both regular and compatible with its grading. We define this filtration as follows: if $A = \oplus A_n$ is a graded abelian group, we set the pth element of the filtration as

$$F^pA := \bigoplus_{n\geq p} A_n \subseteq A. \tag{1}$$

It is clear that that $\{F^pA\}_{p\in\mathbb{Z}}$ is a filtration on A. Furthermore:

$$F^pA \cap A_{p+q} = \Big(\bigoplus_{n\geq p} A_n\Big) \cap A_{p+q} = \begin{cases} A_{p+q} & \text{if } q \geq 0 \\ 0 & \text{if } q < 0 \end{cases}$$

so that

$$\bigoplus_{q\in\mathbb{Z}} (F^pA \cap A_{p+q}) = \bigoplus_{q\geq 0} A_{p+q} = \bigoplus_{n\geq p} A_n = F^pA,$$

and hence the filtration $\{F^pA\}_{p\in\mathbb{Z}}$ is compatible with the natural grading of A. Furthermore, it is clear that for any integer $n \in \mathbb{Z}$, we may take $\mu(n) = n+1$ so that

$$p \geq n+1 \quad \Longrightarrow \quad F^pA \cap A_n = \Big(\bigoplus_{m\geq p} A_m\Big) \cap A_n = \bigoplus_{m\geq p} (A_m \cap A_n) = 0$$

since $A_m \cap A_n = 0$ whenever $m \neq n$. This shows that the above filtration is regular. This filtration is called the *trivial filtration of the graded group A.*

Graded abelian groups arise naturally as complexes so it is important to incorporate differentials into our study of graded abelian groups. Below we give the natural definition of differential in the context of graded abelian groups, which is equivalent to the usual definition of differentials of cochain complexes.

Definition 2. Let $A = \oplus A_n$ be a graded abelian group. An endomorphism $\delta : A \to A$ is called a *differential* if it satisfies:

(i) δ is of degree 1, that is $\delta(A_n) \subseteq A_{n+1}$,

(ii) $\delta \circ \delta = 0$,

Furthermore, if A is filtered by $\{F^pA\}_{p\in\mathbb{Z}}$, we say that δ is *compatible with the filtration* if

(iii) $\delta(F^pA) \subseteq F^pA$.

Remark. Condition (i) tells us that $\delta = \oplus \delta^n$ where $\delta^n : A_n \to A_{n+1}$ is the restriction $\delta^n := \delta|_{A_n}$. Hence a differential $\delta : A \to A$ on a graded abelian group $A = \oplus A_n$ is precisely a cochain complex

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\delta^{n-1}} A_n \xrightarrow{\delta^n} A_{n+1} \longrightarrow \cdots$$

Remark. Suppose A is a graded abelian group with a filtration $\{F^pA\}_{p\in\mathbb{Z}}$ and $\delta : A \to A$ is a compatible differential. Since the filtration is compatible with the grading, we know that each term F^pA of the filtration is itself a graded abelian group. Since $\delta(A_n) \subseteq A_{n+1}$ by assumption and $\delta(F^pA) \subseteq F^pA$ by the above, then the restriction $\delta|_{F^pA}$ sends the nth homogeneous component of F^pA to its $(n+1)$th homogeneous component. Indeed

$$\delta|_{F^pA}((F^pA)_n) = \delta(F^p \cap A_{p+n}) \subseteq \delta(F^pA) \cap \delta(A_{p+n}) = F^pA \cap A_{p+(n+1)} = (F^pA)_{n+1}.$$

so δ restricts to a differential on F^pA.

We collect all these compatibility conditions that we've seen so far. For lack of a better term, we call these groups *admissible* and we fix the following notation:

Definition 3. We say that an abelian group $A = (A, \{A_n\}_{n\in\mathbb{Z}}, \{F^pA\}_{p\in\mathbb{Z}}, \delta)$ is *admissible* if A satisfies the following:

(A.i) A is graded with $A = \oplus A_n$,

(A.ii) A is filtered by $\{F^pA\}_{p\in\mathbb{Z}}$ and this filtration is regular and compatible with the grading,

(A.iii) A has a differential $\delta : A \to A$ that is compatible with the filtration on A.

It is for admissible groups that we can define a cohomology module.

Definition 4. Let A be an admissible group. The *cohomology module* of A is the graded abelian group

$$H^*(A) = \bigoplus_{n\in\mathbb{Z}} H^n(A) \quad \text{where} \quad H^n(A) = \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}}.$$

Remark. For A an admissible group, we can form a cochain complex

$$A_\bullet := \quad \cdots \longrightarrow A_{n-1} \xrightarrow{\delta^{n-1}} A_n \xrightarrow{\delta^n} A_{n+1} \longrightarrow \cdots$$

where $\delta^n := \delta|_{A^n}$ so that $H^n(A)$ is the nth cohomology group of this chain, i.e.

$$H^n(A) = H^n(A_\bullet).$$

Furthermore, we may consider each term F^pA of the filtration as its own cochain complex

$$(F^pA)_\bullet \quad \cdots \longrightarrow (F^pA)_{n-1} \xrightarrow{\delta^{p+n-1}|_{F^pA}} (F^pA)_n \xrightarrow{\delta^{p+n}|_{F^pA}} (F^pA)_{n+1} \longrightarrow \cdots$$

and hence the nth cohomology of this cochain is:

$$H^n((F^pA)_\bullet) = \frac{\ker(\delta^{p+n}|_{F^pA})}{\operatorname{im}(\delta^{p+n-1}|_{F^pA})} = H^{p+n}(F^pA). \tag{2}$$

The cohomology module of A comes equipped with quite a bit of information about the filtered group A. Let us unravel some of it. The inclusion

$$\iota_p : F^pA \hookrightarrow A$$

induces maps $H^n(F^pA) \longrightarrow H^n(A)$ on cohomology for all n and hence we get a map

$$\iota_p^* : H^*(F^pA) \longrightarrow H^*(A).$$

The images of these maps form a filtration of $H^*(A)$:

Lemma 1. *Let A be an admissible group and $H^*(A)$ its cohomology module. Then the images,*

$$F^pH(A) := \iota_p(H^*(F^pA)) \subseteq H^*(A), \qquad p \in \mathbb{Z}$$

form a filtration of $H^*(A)$. This filtration is regular and compatible with the grading of $H^*(A)$.

Proof. Routine. □

Next we discuss the graded group associated to a filtered abelian group.

Definition 5. Let A be a filtered abelian group, with filtration $\{F^pA\}_{p\in\mathbb{Z}}$. The *associated graded group* of A is the graded abelian group

$$\operatorname{gr}(A) := \bigoplus_{n\in\mathbb{Z}} \operatorname{gr}(A)_n \quad \text{where} \quad \operatorname{gr}(A)_n := F^nA/F^{n+1}A.$$

Remark. If furthermore A is filtered and the filtration on A is compatible with its grading, then

$$\begin{aligned} \operatorname{gr}(A)_p &= \frac{\oplus F^pA \cap A_{p+q}}{\oplus F^{p+1}A \cap A_{p+1+q}} \cong \frac{\oplus_q F^pA \cap A_{p+q}}{\oplus_{q'} F^{p+1}A \cap A_{p+q'}} \quad (q' = q+1) \\ &\cong \bigoplus_{q\in\mathbb{Z}} \frac{F^pA \cap A_{p+q}}{F^{p+1}A \cap A_{p+q}} = \bigoplus_{q\in\mathbb{Z}} \frac{(F^pA)_q}{(F^{p+1}A)_{q-1}} \end{aligned}$$

Thus, if we set

$$(F^p\operatorname{gr}(A))_q := (F^pA)_q/(F^{p+1}A)_{q-1}$$

we get that

$$\operatorname{gr}(A) = \bigoplus_{p,q\in\mathbb{Z}} (F^p\operatorname{gr}(A))_q$$

is a *bigraded* abelian group.

2 Spectral Sequences

The following constructions can all be done for any abelian category (most notably the category of chain complexes and the category of sheaves). Here, we will only focus on the category of abelian groups to simplify the exposition.

Definition 6. A (cohomological) *spectral sequence* $\mathbb{E} = (E_r^{p,q}, \delta_r^{p,q})_{r \geq r_0}$ of abelian groups starting at $r = r_0$, consists of the following data: for each $r \geq r_0$ and $p, q \in \mathbb{Z}$ we have

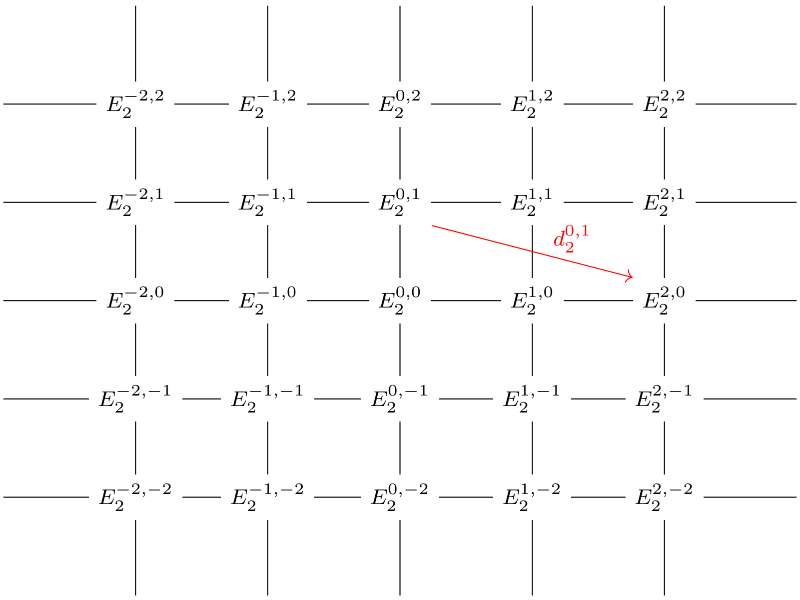
(SS.i) an abelian group $E_r^{p,q}$, called the (p,q)*th term in the rth page of* $\mathbb{E}$,

(SS.ii) a group homomorphism $\delta_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$, called the *rth differential at* (p,q), such that $\delta_r^{p+r,q-r+1} \circ \delta_r^{p,q} = 0$ for all p, q and r,

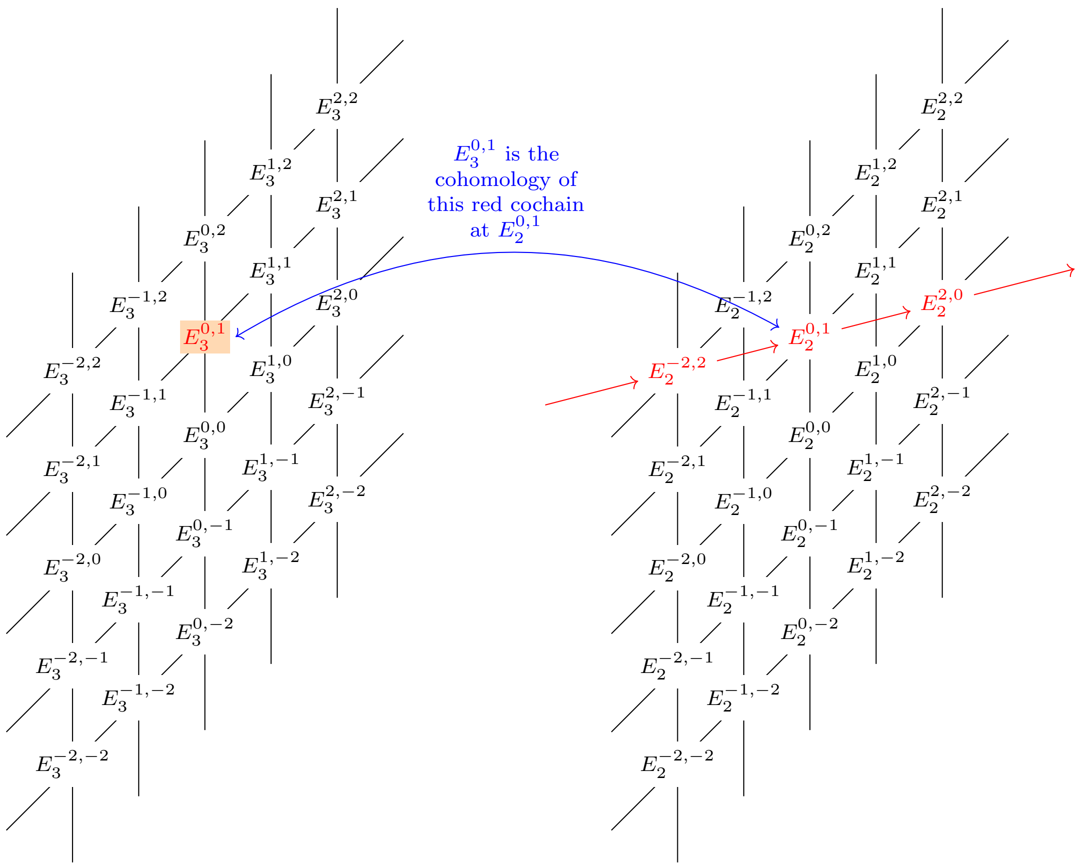
(SS.iii) An isomorphism

$$\frac{\ker \delta_r^{p,q}}{\operatorname{im} \delta_r^{p-r,q+r-1}} \xrightarrow{\sim} E_{r+1}^{p,q}.$$

It is convenient to have a visual representation of spectral sequences. The rth page of a spectral sequence can be viewed as a 2 dimensional grid like $\mathbb{Z} \times \mathbb{Z}$ where we attach $E_r^{p,q}$ to the point with coordinates (p,q). The rth differentials are all diagonal arrows on rth page that go over r places and downwards $(r-1)$ places. For example, the 2nd page of a spectral sequence and its 2nd differentials look like:



To visualize condition (SS.iii), we interpret the isomorphism as an assignment that sends $E_r^{p,q}$ to the cohomology of the cochain induced by the differentials $\delta_2^{p+r,q-r+1}$ and $\delta_2^{p,q}$. Hence we have:



Notice that as long as you have the differentials $d_r^{p,q}$ for every $r \geq r_0$, then given $E_{r_0}^{p,q}$, the rest of the terms $E_r^{p,q}$ are determined. So one can say that a spectral sequence is "determined" by the information on the first page. In general, you need to know all the differentials on every page to reconstruct the spectral sequence from only the first page, but in practice, these higher level differentials are all induced by the differentials on the first page. So most of the time, a spectral sequence is completely by the information on the first page.

Now let us take a closer look at condition (SS.iii). It tells us that $E_{r+1}^{p,q}$ is a quotient of $\ker \delta_r^{p,q} \subseteq E_r^{p,q}$, i.e. a subquotient of $E_r^{p,q}$. Hence, the sequence $E_{r_0}^{p,q}, E_{r_0+1}^{p,q}, E_{r_0+2}^{p,q}, \ldots$ is a sequence of consecutive subquotients. Thus there exists two sequences

$$B_{r_0}^{p,q} \subseteq B_{r_0+1}^{p,q} \subseteq B_{r_0+2}^{p,q} \subseteq \cdots \quad \text{and} \quad Z_{r_0}^{p,q} \supseteq Z_{r_0+1}^{p,q} \supseteq Z_{r_0+2}^{p,q} \supseteq \cdots \tag{3}$$

of subgroups of $E_{r_0}^{p,q}$ that satisfy

(*i*) $B_i^{p,q} \subseteq Z_j^{p,q}$ for all $i, j \geq r_0$,

(*ii*) $E_r^{p,q} \cong Z_r^{p,q}/B_r^{p,q}$,

This is essentially the Correspondence Theorem (cf. Proposition 12). With these observations, we can define an important part of spectral sequences.

Definition 7. Let $\mathbb{E} = (E_r^{p,q}, \delta_r^{p,q})_{r \geq r_0}$ be a spectral sequence of abelian groups starting at $r = r_0$. Then for any $p, q \in \mathbb{Z}$, we define:

- $B_\infty^{p,q} := \bigcup_{r \geq r_0} B_r^{p,q}$,
- $Z_\infty^{p,q} := \bigcap_{r \geq r_0} Z_r^{p,q}$,
- $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$.

Remark. The above definitions immediately generalize to an abelian category $\mathcal{C}$ for which both limits and colimits exist. In this case $B_\infty^{p,q}$, resp. $Z_\infty^{p,q}$, is the colimit, resp. limit, of the family $\{B_r^{p,q}\}_{r \geq r_0}$, resp. $\{Z_r^{p,q}\}_{r \geq r_0}$, of objects of $\mathcal{C}$.

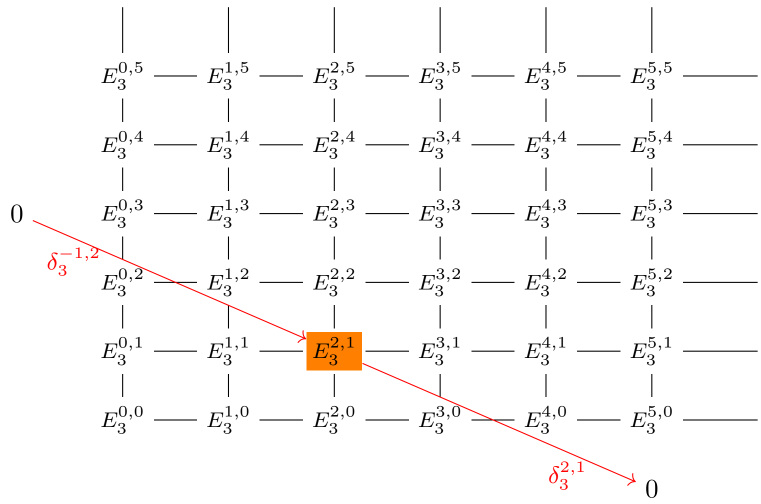
Usually, the sequences $\{B_r^{p,q}\}_{r\geq r_0}$ and $\{Z_r^{p,q}\}_{r\geq r_0}$ in (3) are eventually constant which completely eliminates the need to take limits/colimits. In fact, this stabilization occurs for quite general spectral sequences. Below we introduce a common type of spectral sequence for which the sequences in (3) are eventually constant.

Definition 8. We say that a spectral sequence $\mathbb{E} = (E_r^{p,q}, \delta_r^{p,q})_{r\geq r_0}$ is a *first quadrant* spectral sequence if $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$.

It is clear that for a first quadrant spectral sequence,

$$r \geq \max\{|p|+1, |q|+2\} \quad \Longrightarrow \quad \delta_r^{p,q} = 0 \quad \text{and} \quad \delta_r^{p-r,q+r-1} = 0.$$

The intuition is that if r is large enough, then $\delta_r^{p,q}$ will send $E_r^{p,q}$ to a point outside of the first quadrant and $\delta_r^{p-r,q+r-1}$ will come from some $E_r^{p-r,q+r-1}$ outside of the first quadrant as well.



Furthermore, the conditions $\delta_r^{p,q} = 0$ and $\delta_r^{p-r,q+r-1} = 0$ immediately imply:

$$\begin{aligned}
\delta_r^{p,q} = 0 \quad &\Longrightarrow \quad \ker(\delta_r^{p,q}) = E_r^{p,q} \quad \Longrightarrow \quad E_{r+1}^{p,q} \cong E_r^{p,q}/\mathrm{im}(\delta_r^{p-r,q+r-1}) \quad \Longrightarrow \quad E_r^{p,q} \twoheadrightarrow E_{r+1}^{p,q} \\
\delta_r^{p-r,q+r-1} = 0 \quad &\Longrightarrow \quad \mathrm{im}(\delta_r^{p-r,q+r-1}) = 0 \quad \Longrightarrow \quad E_{r+1}^{p,q} \cong \ker \delta_r^{p,q} \quad \Longrightarrow \quad E_{r+1}^{p,q} \hookrightarrow E_r^{p,q}
\end{aligned}$$

Thus:

$$r \geq \max\{|p|+1, |q|+2\} \quad \Longrightarrow \quad E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots$$

and the sequence stabilizes. Before continuing, we state the special cases when $p = 0$ or $q = 0$ which will become important later on:

$$\begin{aligned}
E_{r_0}^{p,0} \twoheadrightarrow E_{r_0+1}^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_p^{p,0} \twoheadrightarrow E_{p+1}^{p,0} = E_{p+2}^{p,0} = \cdots = E_\infty^{p,0} \quad &\Longrightarrow \quad E_{r_0}^{p,0} \twoheadrightarrow E_\infty^{p,0}, \\
E_\infty^{0,q} = \cdots = E_{q+3}^{0,q} = E_{q+2}^{0,q} \hookrightarrow E_{q+1}^{0,q} \hookrightarrow \cdots \hookrightarrow E_{r_0}^{0,q} \quad &\Longrightarrow \quad E_\infty^{0,q} \hookrightarrow E_{r_0}^{0,q}
\end{aligned} \tag{4}$$

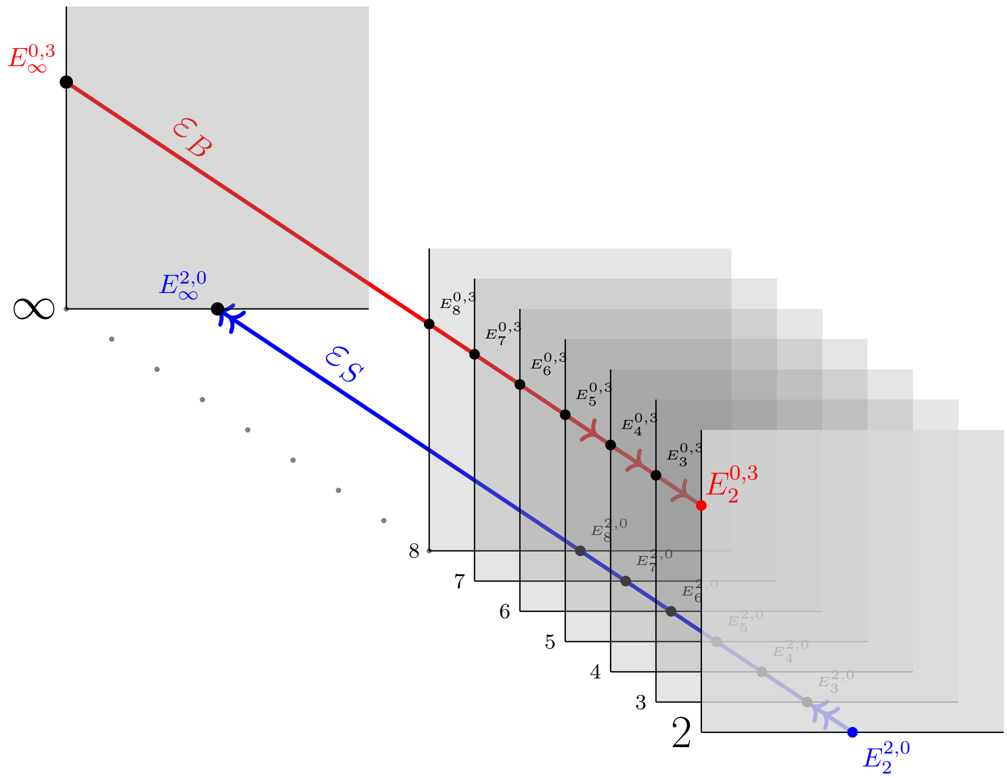
In general, if $\mathbb{E}$ is a 1st quadrant spectral sequence, we have: if $r \geq r_0$ and $p, q > 0$, there are maps

$$\epsilon_r^B : E_r^{p,0} \longrightarrow\!\!\!\!\!\rightarrow E_\infty^{p,0} \qquad\qquad \epsilon_r^S : E_\infty^{0,q} \hookrightarrow E_r^{0,q} \tag{5}$$

defined by composing the morphisms in (4) up to the appropriate page. In fact, we showed that

$$r > p \quad \Longrightarrow \quad \epsilon_r^B \text{ is an isomorphism} \quad , \quad r > q+1 \quad \Longrightarrow \quad \epsilon_r^S \text{ is an isomorphism.} \tag{6}$$

The terms $E_r^{p,0}$ and $E_r^{0,q}$ are called *edge terms* because they lie on the edges of the rth page of $\mathbb{E}$. The maps ϵ_r^B and ϵ_r^S are called the *bottom-edge maps* and *side-edge maps* respectively. Usually, the subindex r is suppressed from the notation since these maps are usually considered only on the first page, i.e. $r = r_0$.



The stabilizing sequences $\{B_r^{p,q}\}_{r\geq r_0}$ and $\{Z_r^{p,q}\}_{r\geq r_0}$ hint at the fact that spectral sequences should converge to something. Below we describe what it means for a spectral sequence to converge.

Definition 9. Let $\mathbb{E} = (E_r^{p,q}, \delta_r^{p,q})_{r\geq r_0}$ be a spectral sequence of abelian groups starting at $r = r_0$. Let $\{A^n\}_{n\in\mathbb{Z}}$ be a family of filtered abelian groups, such that each A^n has a filtration $\{F^pA^n\}_{p\in\mathbb{Z}}$ that satisfies $F^pA^n = 0$ for sufficiently small p and $F^pA^n = A^n$ for sufficiently large. We say that $\mathbb{E}$ *converges to* $\{A^n\}_{n\in\mathbb{Z}}$, denoted by

$$E_{r_0}^{p,q} \Longrightarrow A^{p+q}$$

if for every $p, q \in \mathbb{Z}$, there exists an isomorphism of groups

$$\alpha^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} \mathrm{gr}(A^{p+q})_p = F^pA^{p+q}/F^{p+1}A^{p+q}$$

The required condition on the filtration $\{F^pA^n\}_{p\in\mathbb{Z}}$ of A^n, namely that $F^pA^n = 0$ for sufficiently small p and $F^pA^n = A^n$ for sufficiently large p, can be rephrased as saying that the filtration on A^n is finite, i.e. after possibly reindexing, we have a finite descending chain

$$A^n = F^0A^n \supseteq F^1A^n \supseteq \cdots \supseteq F^pA^n \supseteq 0$$

So the homogenous components of the associated graded group $\mathrm{gr}(A^n)$ are nothing but the composition factors of A^n, i.e.

$$\mathrm{gr}(A^n)_0 = \frac{A^n}{F^1A^n}, \quad \mathrm{gr}(A^n)_1 = \frac{F^1A^n}{F^2A^n}, \quad \cdots \quad \mathrm{gr}(A^n)_p = \frac{F^pA^n}{0} = F^pA^n.$$

Thus we can say that a spectral sequence converges simultaneously to the composition factors of all the A^n's. This is very useful since it follows the general philosophy in algebra that studying composition factors yields a great deal of information about the group itself.

Let us look a little closer. We begin by fixing one of the filtered abelian groups A^n. The terms of the spectral sequence we need to study the decomposition factors of A^n are the terms $E_r^{p,q}$ with $p+q = n$, or more impoprtantly, the terms $E_\infty^{p,q}$. If the spectral sequence is a 1st quadrant spectral sequence, then the relevant terms are $E_\infty^{0,n}, E_\infty^{1,n-1}, \ldots, E_\infty^{n,0}$. Under this assumption, we have that the isomorphisms $\alpha^{p,q}$ give us:

$$E_\infty^{p,n-p} \cong \mathrm{gr}(A^n)_p = p\text{th composition factor of } A^n.$$

Thus, since $\mathbb{E}$ is 1st quadrant, for $p > n$ we have

$$0 = E_\infty^{p,n-p} = \mathrm{gr}(A^n)_p = \frac{F^pA^n}{F^{p+1}A^n} \quad \Longrightarrow \quad F^pA^n = F^{p+1}A^n$$

so the filtration stabilizes precisely at $p = n+1$. Since the filtration is finite, then $F^{n+1}A^n = F^{n+2}A^n = \cdots = 0$. In particular, we have that:

$$E_\infty^{n,0} \cong F^n A^n \hookrightarrow A^n$$

and $E_\infty^{0,n} \cong A^n/F^1A^n$, i.e there is an epimorphism

$$A^n \twoheadrightarrow A^n/F^1A^n \cong E_\infty^{0,n}.$$

If we compose these maps with the edge maps we defined in (5) we obtain: for $r \geq r_0$

$$\varepsilon_r^{n,0} : E_r^{n,0} \longrightarrow A^n \quad \text{and} \quad \varepsilon_r^{0,n} : A^n \longrightarrow E_r^{0,n} \tag{7}$$

Beause of their importance, we record these morphisms:

Definition 10. Let $\mathbb{E} = (E_r^{p,q}, \delta_r^{p,q})_{r \geq r_0}$ be a 1st quadrant spectral sequence of abelian groups starting at $r = r_0$. Suppose that $E_{r_0}^{p,q} \Longrightarrow A^{p+q}$. The morphisms $\varepsilon_r^{n,0}$ and $\varepsilon_r^{0,n}$ defined in (7) are called the *edge maps* of $\mathbb{E}$; the $\varepsilon_r^{0,n}$ are called the *side-edge maps* and the $\varepsilon_r^{n,0}$ are called the *bottom-edge maps.*

The edge maps are very important since they relate the edge terms of $\mathbb{E}$ with the limit groups A^n themselves, and not just their composition factors. This means that the edge maps carry a lot of information about the limit of the spectral sequence. For small p and q we can actually say quite a lot about these edge maps.

Proposition 2. *Let $\mathbb{E} = (E_r^{p,q}, \delta_r^{p,q})_{r \geq r_0}$ be a 1st quadrant spectral sequence of abelian groups starting at $r_0 \leq 2$. Suppose that this spectral sequence converges: $E_r^{p,q} \Longrightarrow A^{p+q}$. Then there is an exact sequence*

$$0 \longrightarrow E_2^{1,0} \xrightarrow{\varepsilon_2^{1,0}} A^1 \xrightarrow{\varepsilon_2^{0,1}} E_2^{0,1} \xrightarrow{\delta_2^{0,1}} E_2^{2,0} \xrightarrow{\varepsilon_2^{2,0}} A^2.$$

Proof. We prove exactness...

(at $E_2^{1,0}$) First, we have that $\varepsilon_2^{1,0}$ is equal to the composition

$$E_2^{1,0} \xrightarrow{\epsilon_2^B} E_\infty^{1,0} \xrightarrow{\alpha^{1,0}} F^1A^1/F^2A^2 = F^1A^1 \hookrightarrow A^1 \tag{8}$$

which is injective since the first arrow is an isomorphism by (6).

(at A^1) The map $\varepsilon_2^{0,1}$ is equal to the composition

$$A^1 \twoheadrightarrow A^1/F^1A^1 = \mathrm{gr}(A^1)_1 \xrightarrow{(\alpha^{0,1})^{-1}} E_\infty^{0,1} \xhookrightarrow{\epsilon_2^S} E_2^{0,1} \tag{9}$$

where the first arrow has kernel F^1A^1, which is the image of $\varepsilon_2^{1,0}$ by (8).

(at $E_2^{0,1}$) The cochain passing through $E_2^{0,1}$ is:

$$\cdots \longrightarrow E_2^{-2,2} \xrightarrow{\delta_2^{-2,2}} E_2^{0,1} \xrightarrow{\delta_2^{0,1}} E_2^{2,0} \longrightarrow \cdots$$

Since $\mathbb{E}$ is 1st quadrant, $E_2^{-2,2} = 0$ and hence $\mathrm{im}(\delta_2^{-2,2}) = 0$. Thus, since $r = 3 \geq \max\{0+1, 1+2\}$, we have that

$$E_\infty^{0,1} \cong E_3^{0,1} \cong \frac{\ker \delta_2^{0,1}}{\mathrm{im}\delta_2^{-2,2}} \cong \ker \delta_2^{0,1}.$$

Since $\epsilon_2^S : E_\infty^{0,1} \hookrightarrow E_2^{0,1}$ is always injective, then by definition of $\varepsilon_2^{0,1}$, or more precisely (9), we have that the image of $\varepsilon_2^{0,1}$ is $\ker \delta_2^{0,1}$.

(at $E_2^{2,0}$) The cochain passing through $E_2^{2,0}$ is:

$$\cdots \longrightarrow E_2^{0,1} \xrightarrow{\delta_2^{0,1}} E_2^{2,0} \xrightarrow{\delta_2^{2,0}} E_2^{4,-1} \longrightarrow \cdots$$

Since $\mathbb{E}$ is 1st quadrant, $E_2^{4,-1} = 0$ so that $\delta_2^{2,0} = 0$ and hence $\ker \delta_2^{2,0} = E_2^{2,0}$. Thus the projection

$$E_2^{2,0} = \ker \delta_2^{2,0} \twoheadrightarrow \frac{\ker \delta_2^{2,0}}{\mathrm{im}\delta_2^{0,1}} \cong E_3^{2,0} \xrightarrow{\epsilon_3^B} E_\infty^{2,0}$$

has kernel $\mathrm{im}\delta_2^{0,1}$ because ϵ_3^B is an isomorphism by (6).

□

3 The Spectral Sequence of a Filtered Abelian Group

We now focus on the main example of interest. Given an admissible group A, e.g. $A = C^*(G, B)$, we will construct a spectral sequence $\mathbb{E}$ that converges to the cohomology module of A. To state this precisely, we recall that the cohomology module $H^*(A)$ is a graded abelian group

$$H^*(A) = \bigoplus_{n\in\mathbb{Z}} H^n(A).$$

By Lemma 1, it is filtered by

$$F^p H^*(A) := \iota_p^*\big(H^*(F^p A)\big)$$

where ι_p^* is the map on cohomology induced by $\iota_p : F^p A \hookrightarrow A$. Since this filtration is compatible with the grading, then each homogeneous component $H^n(A)$ of $H^*(A)$ is filtered by

$$F^p H^n(A) := F^p H^*(A) \cap H^n(A).$$

Since the filtration on $H^*(A)$ is regular, then $F^p H^n(A) = 0$ for sufficiently large p. This means we can talk about convergence of spectral sequences to $\{H^n(A)\}$ since each $H^n(A)$ is a filtered abelian group whose filtration satisfies the necessary finiteness condition. We can now state the main result of this section.

Theorem 3. *Let A be an admissible group. Then there exists a 1st quadrant spectral sequence $\mathbb{E} = (E_r^{p,q}, \delta_r^{p,q})$ starting at $r = 1$ that converges to the cohomology module of A, i.e.*

$$E_1^{p,q} \Longrightarrow H^{p+q}(A).$$

The first page of $\mathbb{E}$ is $E_1^{p,q} = H^{p+q}(\mathrm{gr}(A)_p)$.

The proof is quite technical though not difficult. We fix some notation: let $A = (A, \{A_n\}_{n\in\mathbb{Z}}, \{F^p A\}_{p\in\mathbb{Z}}, \delta)$ be an admissible group. For simplicity we abbreviate:

$$A^p := F^p A.$$

Proof. (of Theorem 3) Let $r \geq 2$. We will carry out the construction of the spectral sequence first and then show that it converges to what we want.

Step 1: Construction of $E_r^{p,q}$.

We will define the groups $E_r^{p,q}$ as quotients $Z_r^{p,q}/B_r^{p,q}$, so let $p, q \in \mathbb{Z}$. The natural projection $A^p/A^{p+r} \to (A^p/A^{p+r})/(A^{p+1}/A^{p+r}) \cong A^p/A^{p+1}$ induces the following map on cohomology:

$$H^{p+q}(A^p/A^{p+r}) \longrightarrow H^{p+q}(A^p/A^{p+1}).$$

So we define $Z_r^{p,q}$ to be the image of this map.

$$Z_r^{p,q} := \mathrm{im}\left(H^{p+q}(A^p/A^{p+r}) \longrightarrow H^{p+q}(A^p/A^{p+1})\right)$$

Next, the natural short exact sequence

$$0 \longrightarrow A^p \hookrightarrow A^{p-r+1} \longrightarrow A^{p-r+1}/A^p \longrightarrow 0 \tag{10}$$

induces a long exact sequence in cohomology with connecting homomorphism Δ:

$$\cdots \longrightarrow H^{p+q-1}(A^{p-r+1}) \longrightarrow H^{p+q-1}(A^{p-r+1}/A^p) \xrightarrow{\Delta} H^{p+q}(A^p) \longrightarrow H^{p+q}(A^{p-r+1}) \longrightarrow \cdots$$

Thus we define $B_r^{p,q}$ as:

$$B_r^{p,q} := \mathrm{im}\Big(\ H^{p+q-1}(A^{p-r+1}/A^p) \xrightarrow{\Delta} H^{p+q}(A^p) \longrightarrow H^{p+q}(A^p/A^{p+1})\ \Big)$$

where the second arrow is the map on cohomology induced by the natural projection $A^p \twoheadrightarrow A^p/A^{p+1}$.

In order to define $E_r^{p,q}$ as $Z_r^{p,q}/B_r^{p,q}$ and have it behave in the same way spectral sequences do, we must show that $Z_{r+1}^{p,q} \subseteq Z_r^{p,q}$, $B_r^{p,q} \subseteq B_{r+1}$ and $B_r^{p,q} \subseteq Z_r^{p,q}$. We prove these three claims below.

Claim: $Z_{r+1}^{p,q} \subseteq Z_r^{p,q}$ for all $r \geq 2$.

The natural projections

$$\begin{array}{ccc} A^p/A^{p+r+1} & \longrightarrow & A^p/A^{p+1} \\ \downarrow & \nearrow & \\ A^p/A^{p+r} & & \end{array}$$

induce on cohomology, the following commutative diagram

$$\begin{array}{ccc} H^{p+q}(A^p/A^{p+r+1}) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) \\ \downarrow & \nearrow & \\ H^{p+q}(A^p/A^{p+r}) & & \end{array}$$

The claim now follows since $Z_{r+1}^{p,q}$ is the image of the top arrow and $Z_r^{p,q}$ is the image of the diagonal arrow.

Claim: $B_r^{p,q} \subseteq B_{r+1}^{p,q}$ for all $r \geq 2$.

The short exact sequence (10) for r and $r+1$ fit side-by-side as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \hookrightarrow & A^{p-r+1} & \twoheadrightarrow & A^{p-r+1}/A^p & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^p & \hookrightarrow & A^{p-r} & \twoheadrightarrow & A^{p-r}/A^p & \longrightarrow & 0. \end{array} \tag{11}$$

Since taking long exact sequences in cohomology is natural, we obtain the following two side-by-side long exact sequences:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^{p+q-1}(A^{p-r+1}) & \longrightarrow & H^{p+q-1}(A^{p-r+1}/A^p) & \xrightarrow{\Delta} & H^{p+q}(A^p) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \| & & \\ \cdots & \longrightarrow & H^{p+q-1}(A^{p-r}) & \longrightarrow & H^{p+q-1}(A^{p-r}/A^p) & \xrightarrow{\Delta} & H^{p+q}(A^p) & \longrightarrow & \cdots \end{array}$$

The claim now follows immediately from the following commutative diagram:

$$\begin{array}{ccccc} H^{p+q-1}(A^{p-r+1}/A^p) & & & & \\ & \overset{\Delta}{\searrow} & & & \\ \downarrow & & H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) \\ & \underset{\Delta}{\nearrow} & & & \\ H^{p+q-1}(A^{p-r}/A^p) & & & & \end{array}$$

since $B_r^{p,q}$ is the image of the top path and $B_{r+1}^{p,q}$ is the image of the bottom path.

Claim: $B_r^{p,q} \subseteq Z_r^{p,q}$ for all $r \geq 2$.

The natural projections

$$\begin{array}{ccc} A^p & \twoheadrightarrow & A^p/A^{p+1} \\ \downarrow & \nearrow & \\ A^p/A^{p+r} & & \end{array}$$

induce the following commutative diagram

$$\begin{array}{ccc} H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) \\ \downarrow & \nearrow & \\ H^{p+q}(A^p/A^{p+r}) & & \end{array} \tag{12}$$

The claim immediately follows by tacking on the connecting homomorphism Δ as follows

$$\begin{array}{ccccc} H^{p+q-1}(A^{p-r+1}/A^p) & \xrightarrow{\Delta} & H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) \\ & & \downarrow & \nearrow & \\ & & H^{p+q}(A^p/A^{p+r}) & & \end{array}$$

and observing that $B_r^{p,q}$ is the image of the top row and $Z_r^{p,q}$ is the the image of the diagonal arrow.

From the three claims we obtain a sequence

$$B_2^{p+q} \subseteq B_3^{p,q} \subseteq \cdots \subseteq Z_3^{p,q} \subseteq Z_2^{p,q}$$

so we may define

$$E_r^{p,q} := Z_r^{p,q}/B_r^{p,q}.$$

This completes the construction of $E_r^{p,q}$ and part (SS.i) of the definition of spectral sequence.

Before moving on, we compute the first page. For $r = 1$, we have $A^{p-r+1}/A^p = A^p/A^p = 0$ and $A^p/A^{p+r} = A^p/A^{p+1}$ so that the maps

$$H^{p+q-1}(A^{p-r+1}/A^p) \xrightarrow{\Delta} H^{p+q}(A^p) \longrightarrow H^{p+q}(A^p/A^{p+1}) \quad \text{and} \quad H^{p+q}(A^p/A^{p+r}) \longrightarrow H^{p+q}(A^p/A^{p+1})$$

are equal to 0 and the identity respectively when $r = 1$. Hence

$$B_1^{p,q} = 0 \quad \text{and} \quad Z_1^{p,q} = H^{p+q}(A^p/A^{p+1}) = H^{p+q}(\mathrm{gr}(A)_p).$$

Step 2: construction of the differentials.

The natural short exact sequence $0 \to A^{p+r} \to A^p \to A^p/A^{p+r} \to 0$ can be placed along side the same short exact sequence for $r = 1$ and then "reduced modulo A^{p+r+1}" to obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{p+r} & \hookrightarrow & A^p & \twoheadrightarrow & A^p/A^{p+r} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & A^{p+1} & \hookrightarrow & A^p & \twoheadrightarrow & A^p/A^{p+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A^{p+1}/A^{p+r+1} & \longrightarrow & A^p/A^{p+r+1} & \longrightarrow & A^p/A^{p+1} & \longrightarrow & 0
\end{array}$$

If we only consider the first and third rows, we obtain two side-by-side long exact sequences in cohomology:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A^p/A^{p+r}) & \xrightarrow{\Delta} & H^{p+q+1}(A^{p+r}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & \overset{\theta}{\searrow} & \downarrow & & \\
\cdots & \longrightarrow & H^{p+q}(A^p/A^{p+r+1}) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) & \xrightarrow{\Gamma} & H^{p+q+1}(A^{p+1}/A^{p+r+1}) & \longrightarrow & \cdots
\end{array} \tag{13}$$

where Γ is the corresponding connecting homomorphism and θ is just the composite map:

$$\theta = \Big(H^{p+q}(A^p/A^{p+r}) \xrightarrow{\Delta} H^{p+q+1}(A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+1}/A^{p+r+1}) \Big)$$

The triangle in (12), with the exponents suitably shifted, appears in the first square of (13) so we can collapse it to get

$$\begin{array}{ccccc}
 & \nearrow & H^{p+q}(A^p/A^{p+r}) & \overset{\theta}{\searrow} & \\
 & & \downarrow & & \\
H^{p+q}(A^p/A^{p+r+1}) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) & \xrightarrow{\Gamma} & H^{p+q+1}(A^{p+1}/A^{p+r+1})
\end{array} \tag{14}$$

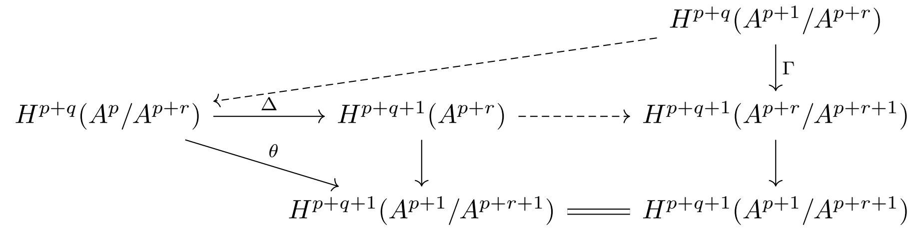
Next, we consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{p+r}/A^{p+r+1} & \hookrightarrow & A^{p+1}/A^{p+r+1} & \twoheadrightarrow & A^{p+1}/A^{p+r} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{p+r}/A^{p+r+1} & \hookrightarrow & A^p/A^{p+r+1} & \twoheadrightarrow & A^p/A^{p+r} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & A^{p+r} & \longrightarrow & A^p & \hookrightarrow & A^p/A^{p+r} & \twoheadrightarrow & 0
\end{array}$$

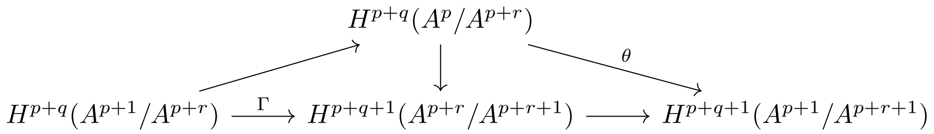
By taking long exact sequences in cohomology, we get

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^{p+q}(A^{p+1}/A^{p+r}) & \xrightarrow{\Gamma} & H^{p+q+1}(A^{p+r}/A^{p+r+1}) & \longrightarrow & H^{p+q+1}(A^{p+1}/A^{p+r+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow & & \\
\cdots & \longrightarrow & H^{p+q}(A^p/A^{p+r}) & \xrightarrow{\Gamma} & H^{p+q+1}(A^{p+r}/A^{p+r+1}) & \longrightarrow & H^{p+q+1}(A^p/A^{p+r+1}) & \longrightarrow & \cdots \\
 & & \| & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & H^{p+q}(A^p/A^{p+r}) & \xrightarrow{\Delta} & H^{p+q+1}(A^{p+r}) & \longrightarrow & H^{p+q+1}(A^p) & \longrightarrow & \cdots
\end{array} \tag{15}$$

Thus if we rotate the first row of the above diagram 90° clockwise and attach it to the right side of (13) we obtain



where the diagonal dashed arrow comes from the left most column of (15) and the horizontal dashed arrow comes from middle column of (15). Rewriting this diagram into a simpler form gives



(16)

Now, we have two diagrams, namely (14) and (16) that are in the form of Lemma 13. Thus we obtain isomorphisms

$$\mathrm{im}\theta \cong \frac{\mathrm{im}\left(H^{p+q}(A^p/A^{p+r}) \to H^{p+q}(A^p/A^{p+1})\right)}{\mathrm{im}\left(H^{p+q}(A^p/A^{p+r+1}) \to H^{p+q}(A^p/A^{p+1})\right)} = \frac{Z_r^{p,q}}{Z_{r+1}^{p,q}}$$

from diagram (14) and

$$\begin{aligned}
\mathrm{im}\theta &\cong \frac{\mathrm{im}\left(H^{p+q}(A^p/A^{p+r}) \xrightarrow{\Delta} H^{p+q+1}(A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+r}/A^{p+r+1})\right)}{\mathrm{im}\left(H^{p+q}(A^{p+1}/A^{p+r}) \xrightarrow{\Gamma} H^{p+q+1}(A^{p+r}/A^{p+r+1})\right)} \\
&\cong \frac{\mathrm{im}\left(H^{p+q}(A^p/A^{p+r}) \xrightarrow{\Delta} H^{p+q+1}(A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+r}/A^{p+r+1})\right)}{\mathrm{im}\left(H^{p+q}(A^{p+1}/A^{p+r}) \xrightarrow{\Delta} H^{p+q+1}(A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+r}/A^{p+r+1})\right)} \\
&= \frac{B_{r+1}^{p+r,q-r+1}}{B_r^{p+r,q-r+1}}
\end{aligned}$$

from diagram (16). Thus we get an isomorphism

$$d_r^{p,q} : \frac{Z_r^{p,q}}{Z_{r+1}^{p,q}} \xrightarrow{\sim} \frac{B_{r+1}^{p+r,q-r+1}}{B_r^{p+r,q-r+1}}.$$

Finally, since $Z_r^{p,q} \supseteq Z_{r+1}^{p,q} \supseteq B_r^{p,q}$, then pojection modulo $Z_{r+1}^{p,q}/B_r^{p,q}$ is a surjective map

$$\pi_r^{p,q} : E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}} \twoheadrightarrow \frac{Z_r^{p,q}}{Z_{r+1}^{p,q}}.$$

Similarly, $B_{r+1}^{p+r,q-r+1} \subseteq Z_{r+1}^{p+r,q-r+1} \subseteq Z_r^{p+r,q-r+1}$ so we have an inclusion map

$$\sigma_{r+1}^{p+r,q-r+1} : \frac{B_{r+1}^{p+r,q-r+1}}{B_r^{p+r,q-r+1}} \hookrightarrow \frac{Z_r^{p+r,q-r+1}}{B_r^{p+r,q-r+1}} = E_r^{p+r,q-r+1}$$

Thus we obtain the differential map

$$\delta_r^{p,q} := \sigma_{r+1}^{p+r,q-r+1} \circ d_r^{p,q} \circ \pi_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{17}$$

which has the required properties because

$$\mathrm{im}\delta_r^{p-r,q+r-1} = \mathrm{im}\sigma_{r+1}^{p,q} = \frac{B_{r+1}^{p,q}}{B_r^{p,q}} \subseteq \frac{Z_{r+1}^{p,q}}{B_r^{p,q}} = \ker \pi_r^{p,q} = \ker \delta_r^{p,q}$$

and hence

$$E_{r+1}^{p,q} = \frac{Z_{r+1}^{p,q}}{B_{r+1}^{p,q}} \cong \frac{Z_{r+1}^{p,q}/B_r^{p,q}}{B_{r+1}^{p,q}/B_r^{p,q}} = \frac{\ker \delta_r^{p,q}}{\mathrm{im}\delta_r^{p-r,q+r-1}}.$$

Step 3: Convergence of the Spectral Sequence.

By Step 1 we have the chains

$$\cdots \subseteq Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \subseteq \cdots \quad \text{and} \quad \cdots \subseteq B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \cdots$$

we define $Z_\infty^{p,q} := \cap Z_r^{p,q}$ and $B_\infty^{p,q} := \cup B_r^{p,q}$. However, since the filtration on A is regular, the above sequences stabilize. Below we construct the candidates for the limits of these sequences and then show that they are indeed the required limits.

Consider (part of) the diagram from Step 2:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{p+r} & \hookrightarrow & A^p & \twoheadrightarrow & A^p/A^{p+r} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & A^{p+1} & \hookrightarrow & A^p & \twoheadrightarrow & A^p/A^{p+1} & \longrightarrow & 0
\end{array}$$

which in cohomology gives us

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A^p/A^{p+r}) & \xrightarrow{\Delta} & H^{p+q+1}(A^{p+r}) & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) & \xrightarrow{\Delta} & H^{p+q+1}(A^{p+1}) & \longrightarrow & \cdots
\end{array} \tag{18}$$

So if we define λ to be the composition

$$\lambda = \left(\; H^{p+q}(A^p/A^{p+r}) \xrightarrow{\Delta} H^{p+q+1}(A^{p+r}) \longrightarrow H^{p+q+1}(A^{p+1}) \; \right)$$

we obtain the diagram

$$\begin{array}{ccccc}
 & \nearrow & H^{p+q}(A^p/A^{p+r}) & \overset{\lambda}{\searrow} & \\
 & & \downarrow & & \\
H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A^p/A^{p+1}) & \xrightarrow{\Delta} & H^{p+q+1}(A^{p+1})
\end{array}$$

Thus Lemma 13 tells us that Δ induces an isomorphism

$$\text{im}\lambda \cong \frac{\text{im}\left(H^{p+q}(A^p/A^{p+r}) \to H^{p+q}(A^p/A^{p+1})\right)}{\text{im}\left(H^{p+q}(A^p) \to H^{p+q}(A^p/A^{p+1})\right)} = \frac{Z_r^{p,q}}{\text{im}\left(H^{p+q}(A^p) \to H^{p+q}(A^p/A^{p+1})\right)}$$

However, by regularity, $\lambda = 0$ for sufficiently large r. More precisely, the cohomology of A^p, as defined in (2), tells us that for $n = p+q+1$ there is an integer $\mu := \mu(p+q+1)$ such that if $p+r \geq \mu(p+q+1)$ then $A^{p+r} \cap A_{p+q+1} = 0$ and thus

$$H^{p+q+1}(A^{p+r}) = H^{q-r+1}((A^{p+r})_\bullet) = \frac{\ker(\delta^{p+q+1}|_{A^{p+r}})}{\text{im}(\delta^{p+q}|_{A^{p+r}})} \subseteq \frac{A^{p+r} \cap A_{p+q+1}}{\text{im}(\delta^{p+q}|_{A^{p+r}})} = 0.$$

Therefore, λ factors through the zero map whenever $r \geq \mu(p+q+1) - p$. Thus we have:

$$r \geq \mu(p+q+1) - p \quad \Longrightarrow \quad Z_r^{p,q} = \text{im}\left(H^{p+q}(A^p) \to H^{p+q}(A^p/A^{p+1})\right)$$

We can now set:

$$Z_\infty^{p,q} := \text{im}\left(H^{p+q}(A^p) \to H^{p+q}(A^p/A^{p+1})\right).$$

As for $B_\infty^{p,q}$, we reduce the short exact sequence $0 \to A^p \to A \to A/A^p \to 0$ modulo A^{p+1} to obtain the following side-by-side short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^p & \longrightarrow & A & \longrightarrow & A/A^p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A^p/A^{p+1} & \longrightarrow & A/A^{p+1} & \longrightarrow & A/A^p & \longrightarrow & 0
\end{array}$$

which induces the following side-by-side long exact sequences in cohomology:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^{p+q-1}(A/A^p) & \longrightarrow & H^{p+q}(A^p) & \longrightarrow & H^{p+q}(A) & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{p+q-1}(A/A^p) & \xrightarrow{\Xi} & H^{p+q}(A^p/A^{p+1}) & \longrightarrow & H^{p+q}(A/A^{p+1}) & \longrightarrow & \cdots
\end{array}$$

So if we define the composition

$$\psi := \Big(\; H^{p+q}(A^p) \longrightarrow H^{p+q}(A) \longrightarrow H^{p+q}(A/A^{p+1}) \; \Big)$$

we obtain the diagram

$$\begin{tikzcd} & H^{p+q}(A^p) \arrow[d] \arrow[dr, "\psi"] & \\ H^{p+q-1}(A/A^p) \arrow[ur] \arrow[r, "\Xi"] & H^{p+q}(A^p/A^{p+1}) \arrow[r] & H^{p+q}(A/A^{p+1}) \end{tikzcd}$$

and thus Lemma 13 implies that we have an isomorphism

$$\mathrm{im}\psi \cong \frac{\mathrm{im}\left(H^{p+q}(A^p) \to H^{p+q}(A^p/A^{p+1})\right)}{\mathrm{im}\left(H^{p+q-1}(A/A^p) \to H^{p+q}(A^p/A^{p+1})\right)} = \frac{Z_\infty^{p,q}}{\mathrm{im}(\Xi)}$$

Now, ψ also appears in the diagram

$$\begin{tikzcd} & H^{p+q}(A^p) \arrow[d, "\iota_p^*"] \arrow[dr, "\psi"] & \\ H^{p+q}(A^{p+1}) \arrow[ur] \arrow[r, "\iota_{p+1}^*"] & H^{p+q}(A) \arrow[r] & H^{p+q}(A/A^{p+1}) \end{tikzcd}$$

where the inclusions $\iota_p : A^p \hookrightarrow A$ define the filtration on $H^*(A)$. Thus the image of the vertical arrow is the $(p+q)$th component of $H^*(A)^p$, and the image of the first horizontal arrow is the $(p+q)$th component of $H^*(A)^{p+1}$. Thus Lemma 13 gives us an isomorphism between these images, i.e. $\mathrm{gr}(H^{p+q}(A))_p$ and the image of ψ. We conclude that

$$\mathrm{gr}(H^{p+q}(A))_p \cong \mathrm{im}\psi \cong \frac{Z_\infty^{p,q}}{\mathrm{im}(\Xi)}$$

Therefore, we will finish the proof of the Theorem when we've established

$$\mathrm{im}(\Xi) = \bigcup_{r \geq 1} B_r^{p,q} = B_\infty^{p,q}. \tag{19}$$

We prove this below.

The three side-by-side exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A^p \arrow[r, hook] \arrow[d, equal] & A^{p-r+1} \arrow[r, two heads] \arrow[d, hook] & A^{p-r+1}/A^p \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & A^p \arrow[r, hook] \arrow[d] & A \arrow[r, two heads] \arrow[d, two heads] & A/A^p \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A^p/A^{p+1} \arrow[r, hook] & A/A^{p+1} \arrow[r, two heads] & A/A^p \arrow[r] & 0 \end{tikzcd}$$

induce on cohomology

$$\begin{tikzcd} \cdots \arrow[r] & H^{p+q-1}(A^{p-r+1}/A^p) \arrow[r, "\Delta"] \arrow[d] & H^{p+q}(A^p) \arrow[r] \arrow[d, equal] & H^{p+q}(A^{p-r+1}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^{p+q-1}(A/A^p) \arrow[r] \arrow[d, equal] & H^{p+q}(A^p) \arrow[r] \arrow[d] & H^{p+q}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^{p+q-1}(A/A^p) \arrow[r, "\Xi"] & H^{p+q}(A^p/A^{p+1}) \arrow[r] & H^{p+q}(A/A^{p+1}) \arrow[r] & \cdots \end{tikzcd}$$

Focusing only on the first two columns, if we combine the equalities and slightly reorder the above diagram, we obtain

$$\begin{tikzcd} & H^{p+q-1}(A^{p-r+1}/A^p) \arrow[dl] \arrow[d, "\Delta"] \\ H^{p+q-1}(A/A^p) \arrow[dr, "\Xi"] & H^{p+q}(A^p) \arrow[d] \\ & H^{p+q}(A^p/A^{p+1}) \end{tikzcd}$$

The image of the composition of the vertical arrows is, by definition, $B_r^{p,q}$. Thus by the commutativity of the diagram we clearly have $B_r^{p,q} \subseteq \mathrm{im}(\Xi)$ for all r and hence

$$\bigcup_{r\geq 1} B_r^{p,q} \subseteq \mathrm{im}(\Xi).$$

□

4 The Hochschild-Serre Spectral Sequence

Let B be a G-module. Then we define the n-cochains of B as

$$C^n(G,B) := \{f : \underbrace{G \times \cdots \times G}_{n \text{ times}} \to B \mid f \text{ is continuous and normalized}\}.$$

Here, we say that $f : G^n \to A$ is *normalized* if $f(g_1, \ldots, g_n) = 0$ whenever $g_i = 1$ for at least one $i = 1, \ldots, n$.

Since B is an abelian group, pointwise addition makes $C^n(G,B)$ into an abelian group. For convenience, we set $G^0(G,B) = \{0\}$; notice that this implies we can set $C^0(G,B) = B$. The cochains of B form a cochain complex as follows: for $n \geq 0$ we have the *nth differential*

$$\delta^n : C^n(B,G) \longrightarrow C^{n+1}(B,G)$$

defined as:

$$\delta^n(f)(g_1, \ldots, g_{n+1}) = g_1.f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n). \quad (20)$$

Notice that if f is normalized, then so is $\delta^n f$. Then, it is routine, but tedious, to verify the following properties.

Proposition 4. *Let B be a G-module, then each δ^n is a group homomorphism and they satisfy $\delta^{n+1} \circ \delta^n = 0$, i.e. we have the cochain complex:*

$$0 \longrightarrow B \xrightarrow{\delta^0} C^1(G,B) \xrightarrow{\delta^1} C^2(G,B) \xrightarrow{\delta^2} C^3(G,B) \longrightarrow \cdots$$

With this cochain comples we can define cohomology groups as usual:

$$Z^n(G,B) := \ker \delta^n, \quad B^n(G,B) := \mathrm{im}(\delta^{n-1}) \quad \text{and} \quad H^n(G,B) := Z^n(G,B)/B^n(G,B).$$

The elements of $Z^n(G,B)$, resp. $B^n(G,B)$, are called n-cocycles, resp. n-coboundaries.

Remark. For $n = 0$ and $n = 1$, the defining formula for δ^n in (20) yields the following well known descriptions of the low level cocyles and coboundaries:

$$Z^0(G,B) = B^G, \quad B^0(G,B) = 0 \quad \text{and} \quad H^0(G,B) = B^G;$$
$$Z^1(G,B) = \{f : G \to B \mid f(gh) = f(g) + g.f(h)\},$$
$$B^1(G,B) = \{f : G \to B \mid \exists b \in B,\ f(g) = g.b - b\}$$

From this data we can form the *total complex* of B:

$$C^*(G,B) := \bigoplus_{n=0}^{\infty} C^n(G,B)$$

which is automatically a graded abelian group whose nth homogeneous component is simply $C^*(G,B)_n = C^n(G,B)$. Furthermore, the differentials combine to give us

$$\delta := \underset{n\geq 0}{\oplus} \delta^n : C^*(G,B) \longrightarrow C^*(G,B) \quad \text{defined by} \quad \delta\big((f^n)_{n\geq 0}\big) = (\delta^n f^n)_{n\geq 0}. \quad (21)$$

From the properties of each δ^n in Proposition 4, we immediately obtain that δ is an endomorphism of $C^*(G,B)$ of *degree 1*, i.e. it sends homogenous elements of degree n to homogeneous elements of degree $n+1$. Furthermore, we can form the *cohomology module* as

$$H^*(G,B) := \bigoplus_{n\in\mathbb{Z}} H^n(G,B).$$

Remark 5. If we remove the assumption that our cocycles be normalized, we obtain usual group cohomology. However, these two cohomologies are isomorphic so there is no loss in generality to assume that our cocylces are normalized.

Finally, we need to define a filtration on $C := C^*(G, B)$. Let $K \trianglelefteq G$ be a normal subgroup of G. We define a filtration of C starting at $p = 0$,

$$C = F^0C \supseteq F^1C \supseteq F^2C \supseteq \cdots$$

where each piece of the filtration F^pC is defined as the following graded group:

$$F^pC = \bigoplus_{n=0}^{\infty} F^pC^n$$

where $F^pC^n \subseteq C^n := C^n(G, B)$ is defined as follows:

$$F^pC^n := \begin{cases} \{f \in C^n \mid f \text{ factors through } \underbrace{G \times \cdots \times G}_{n-p\,\text{times}} \times \underbrace{G/K \times \cdots \times G/K}_{p\,\text{times}}\} & 1 \le p \le n \\ 0 & p > n. \end{cases}$$

For example, if $p = 1$, then $F^1C = \oplus F^1C^n$ where

$$F^1C^n = \begin{cases} \{f \in C^n \mid f \text{ factors through } G \times \cdots \times G \times G/K\} & \text{if } n \ge 1 \\ 0 & \text{if } n = 0 \end{cases}$$

Thus F^1C consists of all cocycles f whose value $f(\gamma_1, \ldots, \gamma_n)$ depends only on $\gamma_1, \ldots, \gamma_{n-1}$ and the coset $\gamma_n K$.

In general, F^pC is the set of cocycles f whose value $f(\gamma_1, \ldots, \gamma_n)$ depend only on $\gamma_1, \ldots, \gamma_{n-p}$ and the cosets $\gamma_{n-p+1}K, \ldots, \gamma_n K$.

Proposition 6. *The filtration defined above is regular and compatible with the grading of $C^*(G, B)$ and its differential δ. Thus $C^*(G, B)$ is an admissible group.*

Proof. By definition of the filtration as $F^pC = \oplus F^pC^n$, the filtration is compatible with the grading. The filtration is also clearly regular, since if $p > n$, then $F^pC \cap C^n = F^pC^n = 0$ by definition.

Showing that the filtration is compatible with the differential is more tedious so we only illustrate the argument with an example. Suppose that $f \in F^2C^2 \subset C^2$. Then

$$\delta f(g_1, g_2, g_3) = g_1 f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) \tag{22}$$

Next, lets choose different representatives g_2' and g_3' of the cosets g_2K and g_3K respectively. We must prove that $(\delta f)(g_1, g_2, g_3) = (\delta f)(g_1, g_2', g_3')$. Since $f \in F^2C^2$, then f factors through $(G/K) \times (G/K)$ and thus

$$\begin{aligned} f(g_2, g_3) &= f(g_2K, g_3K) = f(g_2'K, g_3'K) = f(g_2', g_3') \\ f(g_1g_2, g_3) &= f(g_1g_2K, g_3K) = f(g_1g_2'K, g_3'K) = f(g_1g_2', g_3') \\ f(g_1, g_2g_3) &= f(g_1, g_2g_3K) = f(g_1, g_2'g_3'K) = f(g_1, g_2'g_3') \\ f(g_1, g_2) &= f(g_1, g_2K) = f(g_1, g_2'K) = f(g_1, g_2'). \end{aligned}$$

Therefore the right hand side of (22) depends only of g_1, g_2K and g_3K, that is $\delta f \in F^2C^3 \subset F^2C$. □

Remark. The filtration defined above is not compatible with the cup product of cocycles. More precisely, the cup product $\cup : C^p(G, B) \times C^q(G, B') \to C^{p+q}(G, B \otimes B')$ of cocyles of two G-modules B and B' is defined by

$$(f \cup g)(\gamma_1, \ldots, \gamma_{p+q}) := f(\gamma_1, \ldots, \gamma_p) \otimes \gamma_1 \cdots \gamma_p \cdot g(\gamma_{p+1}, \ldots, \gamma_{p+q})$$

and they induce a cup product on cohomology, however, for the filtration defined above, in general we have

$$F^pC(G, B) \cup F^qC(G, B') \not\subseteq F^{p+q}C(G, B \otimes B')$$

A different filtration is needed for the compatibility of the cup product to hold. One defines the new filtration

$$F^{*p}C^n := \begin{cases} \{f \in C^n \mid \text{if } n - p + 1 \text{ many } g_i\text{'s lie in } K, \text{ then } f(g_1, \ldots, g_n) = 0\ \} & 1 \le p \le n \\ 0 & p > n. \end{cases}$$

It is obvious that $F^{*p}C^n \subseteq F^pC^n$ and these inclusions induce maps $H^n(F^{*p}C) \to H^n(F^pC)$. One can show, with a bit of work, that these induced maps are isomorphisms and that they induce isomorphisms between the two spectral sequences that these filtrations each define via Theorem 3. While this new filtration turns out to be compatible with the cup product, it is less amenable to spectral sequence computations, which is why both filtrations are needed. See §1 of Chapter II of [HS53] for more details.

Now that we have a filtration on $C^*(G,B)$, we have a spectral sequence. By Theorem 3, there is a 1st quadrant spectral sequence, starting at $r_0 = 1$, converging to the cohomology module $H^*(G,B)$. Below we compute the first and second pages of the spectral sequence so we can compute the short exact sequence of edge terms from Proposition 2.

Given $p, q \geq 0$, then F^pC^{p+q} consists of cocyles $f \in C^{p+q}(G,B)$ whose values $f(\gamma_1, \ldots, \gamma_{p+q})$ depend only on $\gamma_1, \ldots, \gamma_q$ and the cosets $\gamma_{q+1}, \ldots, \gamma_{p+q}$. Thus we can restrict the first q arguments to K to obtain a map $(G/K)^p \to C^q(K,B)$. We describe this map more precisely below.

Given a fixed choice of coset representatives for G/K, we have a (set theoretic) section $G/K \to G$ which we denote by $x \mapsto x^*$. We only require that the identity $K \in G/K$ lifts to the identity in G, i.e. $K^* = 1$. With this choice of representatives we can define

$$r_p : F^pC^{p+q} \longrightarrow C^p(G/K, C^q(K,B)) \quad \text{with} \quad f \mapsto r_pf : \underbrace{G/K \times \cdots \times G/K}_{p\,\text{times}} \to C^q(K,B)$$

where given $x_1, \ldots, x_p \in G/K$, the cochain $r_pf(x_1, \ldots, x_p) \in C^p(K,B)$ is defined as:

$$r_pf(x_1, \ldots, x_p)(k_1, \ldots, k_q) = f(k_1, \ldots, k_q, x_1^*, \cdots, x_p^*).$$

Since the last p arguments of $f \in F^pC^{p+q}$ depend only on their class modulo K, then r_pf clearly does not depend on the the section $x \mapsto x^*$. Furthermore, since we chose $K^* = 1$, then r_pf is also normalized. The definition of r_p is level-wise so it immediately upgrades to a homomorphism on F^pC.

Next, we show that r_p factors through $F^{p+1}C \subseteq F^pC$. If $f \in F^{p+1}C$ of level $p+1+q$, for some $q \geq 0$, then we can view f as an element of $F^{p+1}C^{(p+1)+q} \subset F^pC^{p+q+1}$. Thus, for any $k_1, \ldots, k_{q+1} \in K$ and $x_1, \ldots, x_p \in G/K$, we have

$$r_pf(x_1, \ldots, x_p)(k_1, \ldots, k_{q+1}) = f(k_1, \ldots, k_q, k_{q+1}, x_1^*, \ldots, x_p^*).$$

Notice that the values of $f \in F^{p+1}C^{p+q+1}$ depend on the first q arguments and the cosets of the last $p+1$ arguments. In particular, f depends on the coset $k_{q+1}K = K$ so we can substitute k_{q+1} with 1 to obtain

$$r_pf(x_1, \ldots, x_p)(k_1, \ldots, k_{q+1}) = f(\ldots, 1, \ldots) = 0$$

since f is normalized. Thus the restriction map r_p induces a homomorphism

$$\overline{r_p} : \mathrm{gr}(C^*(G,B))_p = F^pC/F^{p+1}C \longrightarrow C^p(G/K, C^*(G,B)).$$

Next, we show that these restriction maps r_p commute with the differentials. More precisely, if we write the differentials of $C^*(G/K, C^*(K,B))$ and $C^*(K,B)$ as $\delta_{G/K}$ and δ_K respectively, then we have the following lemma.

Lemma 7. *Given $f \in F^pC$, then for any $x_1, \ldots, x_p \in G/K$, we have*

$$r_p(\delta_{G/K}f)(x_1, \ldots, x_p) = \delta_K(r_pf(x_1, \ldots, x_p)).$$

In particular, $\overline{r_p}$ induces a homomorphism on cohomology

$$\overline{r_p}^* : H^{p+q}(F^pC/F^{p+1}C) \longrightarrow C^p(G/K, H^q(K,B)).$$

Proof. Follows from the definition of the coboundary maps and the definition of F^pC. Maybe I'll add a proof later... □

In fact, more is true

Theorem 8. *(Hochschild-Serre) The map $\overline{r_p}^*$ above is an isomorphism.*

Thus, if $\mathbb{E} = (E_r^{p,q})_{r\geq 1}$ is the spectral sequence attached to the filtration F^pC, given by Theorem 3, then

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}(C^*(G,B))_p) = H^{p+q}(F^pC/F^{p+1}C) \cong C^p(G/K, H^q(K,B))$$

Since passing to the second page is just "taking cohomology"*, then we have

$$E_2^{p,q} \cong H^p(G/K, H^q(K,B)).$$

Thus we can compute the exact sequence of edge terms.

*This requires much more work since one must also compute the differentials on the first page, which Hochschild and Serre do in §4 of Chapter II of [HS53]

Corollary 9. *We have the following exact sequence,*

$$0 \longrightarrow H^1(G/K, B^K) \xrightarrow{\text{inf}} H^1(G,B) \xrightarrow{\text{res}} H^1(K,B)^{G/K} \xrightarrow{t} H^2(G/K, B^K) \longrightarrow H^2(G,B)$$

where inf *and* res *are the usual inflation and restriction maps.*

Remark. The map t is called *transgression* and it is the differential map $\delta_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$ of the spectral sequence attached to the Hochschild-Serre filtration on $C^*(G,B)$. The proof of Theorem 3 gives an explicit description of the differentials because the construction of these differentials is based on Lemma 13 which tells you precisely what the isomorphism is. Let us work though these definitions.

Set $r = 2$, $p = 0$ and $q = 1$. In the proof of Theorem 3, cf. equation (17), we defined $\delta_2^{0,1}$ as the composition

$$E_2^{0,1} = \frac{Z_2^{0,1}}{B_2^{0,1}} \xrightarrow{\text{mod } Z_3^{0,1}/B_2^{0,1}} \frac{Z_3^{0,1}}{Z_2^{0,1}} \xrightarrow{d_2^{0,1}} \frac{B_3^{2,0}}{B_2^{2,0}} \hookrightarrow \frac{Z_2^{2,0}}{B_2^{2,0}} = E_2^{2,0}$$

$$\vdots$$

We arrive at the following description of the transgression map... (incomplete)

Example. We can use the Hochschild-Serre spectral sequence to compute the cohomology of dihedral groups quite easily. Here we just compute the cohomology with coefficients in $\mathbb{Z}$ (with the trivial action). This can be done easily since the cohomology of cyclic groups can be computed directly from the derived functors approach; in fact, there is an explicit free resolution of $\mathbb{Z}$ that is 2-periodic. Here we just quote the computation: let C_n be a cyclic group of order n, then

$$H^n(C, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{C_n} = \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}[n] = 0 & \text{if } n \text{ is odd} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } n \text{ is even} \end{cases}$$

Lets consider the case when n is odd, and the dihedral group $G = \mathbb{D}_{2n}$. We know that G, is the extension

$$1 \longrightarrow C_n \longrightarrow \mathbb{D}_{2n} \longrightarrow C_2 \longrightarrow 1.$$

Thus the exact sequence of edge terms for $G = \mathbb{D}_{2n}$ and $K = C_n$ is

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^1(C_2,\mathbb{Z}) & \longrightarrow & H^1(\mathbb{D}_{2n},\mathbb{Z}) & \longrightarrow & H^1(C_n,\mathbb{Z})^{C_2} & \longrightarrow & H^2(C_2,\mathbb{Z}) & \longrightarrow H^2(\mathbb{D}_{2n},\mathbb{Z}) \\ & \| & & \| & & \| & & \| & \\ 0 \longrightarrow & 0 & \longrightarrow & H^1(\mathbb{D}_{2n},\mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow H^2(\mathbb{D}_{2n},\mathbb{Z}) \end{array}$$

5 The Euler Characteristic of a Group

Classically, the Euler characteristic of a topological space X is given by the alternating sum

$$\chi(X) := \sum_{i=0}^{\infty} (-1)^i \text{rk}(H_i^{\text{sing}}(X))$$

where each term $\beta_i := \text{rk}(H_i^{\text{sing}}(X))$ is the ith Betti number. This gives us a template to define the Euler characteristic of a group.

Firstly, given an abelian group A, we define the *rank* of A as the cardinality of a maximal $\mathbb{Z}$-linearly independent subset of A. This definition is not very actionable, so we describe it in a different way. Let A_{tor} denote the torsion subgroup of A. Since A/A_{tor} is a torsion-free $\mathbb{Z}$-module ($\mathbb{Z}$ being a PID), it is a free abelian group, so $A/A_{\text{tor}} \cong \mathbb{Z}^r$ for $r = \text{rk}(A)$. Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, then tensoring the short exact sequence $0 \to A_{\text{tor}} \to A \to A/A_{\text{tor}} \to 0$ with $\mathbb{Q}$ yields the short exact sequence

$$0 \longrightarrow \underbrace{A_{\text{tor}} \otimes_{\mathbb{Z}} \mathbb{Q}}_{=0} \longrightarrow A \otimes_Z \mathbb{Q} \longrightarrow \underbrace{A/A_{\text{tor}} \otimes_Z \mathbb{Q}}_{\cong \mathbb{Q}^r} \longrightarrow 0$$

Thus $\text{rk}(A) = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$. This argument also shows that the rank of an abelian group is *additive*, that is, if

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is a short exact sequence of abelian groups, then

$$\text{rk}(A') - \text{rk}(A) + \text{rk}(A'') = 0$$

since the same is true for vector spaces. Furthermore, if we have a finite filtration $A = A^0 \supseteq A^1 \supseteq \cdots \supseteq A^n \supseteq 0$, then repeated applications of this principal gives

$$\begin{aligned}
\text{rk}(A) &= \text{rk}(A^0) \\
&= \text{rk}(A^0/A^1) + \text{rk}(A^1) \\
&= \text{rk}(A^0/A^1) + \text{rk}(A^1/A^2) + \text{rk}(A^2) \\
&= \text{rk}(A^0/A^1) + \text{rk}(A^1/A^2) + \text{rk}(A^2/A^3) + \text{rk}(A^3)
\end{aligned}$$

Continuing in this manner, one concludes that

$$\text{rk}(A) = \sum_{i=0}^{n} \text{rk}(A^i/A^{i+1}) = \sum_{i=0}^{n} \text{rk}\left(\text{gr}_p(A)\right) \tag{23}$$

This motivates the following definition.

Definition 11. Let $A = \oplus A_n$ be a graded abelian group such that each A_n is finitely generated. The *Poincaré series* attached to A is defined as the formal power series

$$P(A,t) := \sum_{n=0}^{\infty} \text{rk}(A_n) t^n.$$

The *Euler characteristic* of A is defined as

$$\chi(A) := P(A,-1) = \sum_{n=0}^{\infty} (-1)^n \text{rk}(A_n)$$

whenever this series converges.

Remark. If $E = \oplus E^{p,q}$ is a bigraded abelian group, we extend the definition of Poincaré series and Euler characteristic as follows:

$$P(E,t) := \sum_{n=0}^{\infty} (-1)^n \text{rk}\left(\bigoplus_{p+q=n} E^{p+q}\right) \quad \text{and} \quad \chi(E) = P(E,-1).$$

Example. If $A = H_*(X)$ is the singular homology of a space X, then we recover the usual definition of the Euler characteristic of a topological space.

Example. Let $A = \mathbb{Q}[x]$, then $A_n = \mathbb{Q}x^n$ as rank 1. Therefore

$$P(A,t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t},$$

However, the sum $P(A,-1)$ does not converge, though one could facetiously say that $\chi(\mathbb{Q}[x]) = 1/2$.

Usually, we will be interested in computing the Euler characteristic of differential graded groups. This extra structure is essential for our computations. The key observation in this regard is the following lemma.

Lemma 10. *Let $A = \oplus A_n$ be a graded abelian group with a differential $\delta : A \to A$. Then*

$$\chi(A) = \chi(H^*(A)).$$

Proof. For each $n \geq 1$, the differential $\delta^n : A_n \to A_{n+1}$ induces an isomorphism $A_n/\ker \delta^n \cong \operatorname{im} \delta^n$. Since the rank is additive, then

$$\text{rk}(A_n) = \text{rk}(\ker \delta^n) + \text{rk}(\operatorname{im} \delta^n).$$

Furthermore, $H^n(A) = \ker \delta^n / \operatorname{im} \delta^{n-1}$, so

$$\text{rk}(\ker \delta^n) = \text{rk}(H^n(A)) + \text{rk}(\operatorname{im} \delta^{n-1})$$

If we plug this into the the first equation, we get

$$\text{rk}(A_n) = \text{rk}(H^n(A)) + \text{rk}(\operatorname{im} \delta^{n-1}) + \text{rk}(\operatorname{im} \delta^n).$$

Therefore

$$\begin{aligned}\chi(A) &= \sum_{n=0}^{\infty}(-1)^n \mathrm{rk}(A_n) \\ &= \sum_{n=0}^{\infty}(-1)^n \big(\mathrm{rk}(H^n(A)) + \mathrm{rk}(\mathrm{im}\,\delta^{n-1}) + \mathrm{rk}(\mathrm{im}\,\delta^n)\big) \\ &= \chi(H^*(A)) + \sum_{n=0}^{\infty}(-1)^n \big(\mathrm{rk}(\mathrm{im}\,\delta^{n-1}) + \mathrm{rk}(\mathrm{im}\,\delta^n)\big) \\ &= \chi(H^*(A))\end{aligned}$$

since the left over series is telescopic. □

This Lemma gives us a way to compute $\chi(A)$ using spectral sequences as follows. Suppose we have an admissible group A and thus a spectral sequence $\mathbb{E} = (E_r^{p,q})$ converging to $H^*(A)$. Fix a positive integer n. Since the spectral sequence is 1st quadrant, then we know that the filtration on $H^n(A)$ is finite and of the form

$$H^n(A) \supseteq H^n(A)^1 \supseteq \cdots \supseteq H^n(A)^n \supseteq 0,$$

and the convergence of the spectral sequence tells us that

$$E_\infty^{p,n-p} \cong \mathrm{gr}(H^n(A))_p.$$

Thus equation (23) implies

$$\mathrm{rk}\Big(\bigoplus_{p+q=n} E_\infty^{p,q}\Big) = \sum_{p=0}^{n} \mathrm{rk}(E_\infty^{p,n-p}) = \sum_{p=0}^{n} \mathrm{rk}(\mathrm{gr}(H^n(A))_p) = \mathrm{rk}(H^n(A))$$

and thus

$$\chi(A) = \chi(H^*(A)) = \sum_{n=0}^{\infty}(-1)^n \mathrm{rk}(H^n(A)) = \sum_{n=0}^{\infty}(-1)^n \mathrm{rk}\Big(\bigoplus_{p+q=n} E_\infty^{p,q}\Big) = \chi(E_\infty)$$

where E_∞ is the bigraded abelian group $\oplus_{p,q\in\mathbb{Z}} E_\infty^{p,q}$. Furthermore, since the spectral sequence is 1st quadrant, then we know that $E_r^{p,q} \cong E_\infty^{p,q}$ for r sufficiently large, thus

$$r > \max\{p, q+1\} \quad \Longrightarrow \quad \chi(E_r) = \chi(E_{r+1}) = \cdots = \chi(E_\infty) = \chi(A).$$

We record this result.

Proposition 11. *Let A be an admissible group. Let $\mathbb{E} = (E_r^{p,q})$ be a 1st quadrant spectral sequence converging to $H^*(A)$. Then*

Miscellaneous

Proposition 12. *Let M be an R-module. Suppose that $M = M_0, M_1, M_2, \ldots$ is a sequence of succesive subquotients, that is M_{n+1} is a subquotient of M_n. Then there exist two sequences $B_1, B_2, \ldots$ and $Z_1, Z_2, \ldots$ of submodules of M such that*

(i) $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$,

(ii) $Z_1 \supseteq Z_2 \supseteq Z_3 \subseteq \cdots$,

(iii) $B_i \subseteq Z_j$ *for any* $i, j \geq 0$,

(iv) $M_i \cong Z_i/B_i$.

Proof. Since M_1 is a subquotient of M, then there are submodules, say B_1 and Z_1, of M such that $B_1 \subseteq Z_1$ and $M_1 \cong Z_1/B_1$. Next, since M_2 is a subquotient of M_1, there exists submodules $Z_2', B_2' \subseteq M_1$ such that $M_2 \cong Z_2'/B_2'$. By the Correspondence Theorem, these two correspond to submodules Z_2 and B_2 of Z_1 that contain B_1, i.e. $Z_2' = Z_2/B_1$, $B_2' = B_2/B_1$ and $M_2 \cong Z_2'/B_2' = (Z_2/B_1)/(B_2/B_1) \cong Z_2/B_2$. Thus we have

$$B_1 \subseteq B_2 \subseteq Z_2 \subseteq Z_1 \subseteq M \quad \text{and} \quad M_1 \cong Z_1/B_2, \quad M_2 \cong Z_2/B_2.$$

Continuing in this manner, we obtain the proposition. □

Lemma 13. *Suppose we have a commutative diagram of abelian groups*

$$\begin{array}{ccccc} & & C & & \\ & \nearrow & \downarrow{\scriptstyle \phi} & \searrow^{\psi} & \\ A' & \xrightarrow{\phi'} & A & \xrightarrow{\eta} & A'' \end{array}$$

whose bottom row is exact. Then η induces an isomorphism $\mathrm{im}\phi/\mathrm{im}\phi' \xrightarrow{\sim} \mathrm{im}\psi$.

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