Étale Algebras

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1 Étale algebras

Let k be a field and choose some algebraic closure $\overline{k}$ of k. The n-fold cartesian product $k \times \cdots \times k$ can be made into a k-algebra via the diagonal embedding.

Definition 1. (Bourbaki) Let A be a k-algebra. Then

1. A is *diagonalizable* if $A \cong k \times \cdots \times k$ as k-algebras.
2. A is *étale* if $A \otimes_k K$ is a diagonalizable K-algebra for some field K containing k.

Next we focus on the case that A is a *finite* k-algebra, that is, A is a finite dimensional k-vector space. Finite algebras are particularly nice. An immediate observation is that A has only finitely many maximal ideals. Indeed, if S is any finite set of maximal ideals of A, then the Chinese Remainder Theorem tells us that

$$\Phi : A \longrightarrow \prod_{\mathfrak{m} \in S} A/\mathfrak{m} \quad \text{defined by} \quad a \mapsto (a + \mathfrak{m})_{\mathfrak{m} \in S}$$

is surjective and hence

$$\dim_k A \geq \dim_k \prod_{\mathfrak{m} \in S} A/\mathfrak{m} = \sum_{\mathfrak{m} \in S} \dim_k A/\mathfrak{m} \geq \sum_{\mathfrak{m} \in S} 1 = |S|.$$

in other words, $|\operatorname{Specm}(A)| \leq \dim_k A < \infty$.

With this observation, we can compute the nilradical of A and compute the reduced ring A/N.

Lemma 1. *Let A be a finite k-algebra with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$, and let N be the nilradical of A. Then*

$$N = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r.$$

In particular, A/N is isomorphic to a product of r fields.

Proof. By Zorn's lemma, we know that N is the intersection of all prime ideals of A, hence $N \subseteq \cap \mathfrak{m}_i$, so we only need to prove the reverse inclusion which amounts to Hilbert's Nullestellensatz.

Since A is a finitely generated k-algebra, then $A \cong k[x_1, \ldots, x_n]/I$ for some proper ideal $I \subset k[x_1, \ldots, x_n]$ so we identify A with this quotient. Let $f \in k[x_1, \ldots, x_n]$ be such that $f + I \in A$ is in the intersection of all the maximal ideals of A, that is

$$f \in \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}.$$

We want to show that $f + I$ is in N, i.e. there exists a positive integer n such that $(f + I)^n = 0$ or equivalently, $f^n \in I$.

By the weak Nullstellensatz, we have that the algebraic set

$$V(I) = \{p = (p_1, \ldots, p_n) \in \overline{k}^n \mid g(p) = 0, \ \forall g \in I\}$$

is non-empty, so let $b \in V(I)$. The image R of the evaluation map

$$\epsilon_b : k[x_1, \ldots, x_n] \longrightarrow \overline{k} \quad \text{defined by} \quad g \mapsto g(b)$$

is a subgring of $\overline{k}$. In fact, R is finite dimensional over k since ϵ_b factors through a surjective k-linear map $A \to R$ where A is finite dimensional by assumption. This implies that the image of ϵ_b is a field. Indeed, if $\alpha \in R \setminus \{0\}$, then the multiplication map $x \mapsto \alpha x$ is an injective k-linear endomorphism of R (notice that R is an integral domain by being a subring of a field), and hence an isomorphism which yields a multiplicative inverse for α.

Therefore, $\ker \epsilon_b$ is a maximal ideal of $k[x_1, \ldots, x_n]$ that contains I. By our choice of f, we have that $f \in \ker \epsilon_b$. We have shown that for every $b \in V(I)$, we have $f(b) = 0$. By the Nullstellensatz, there is a positive integer m for which $f^n \in I$, as required. ∎

Corollary 2. *Let A be a finite k-algebra with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. Then there is a positive integer n such that*

$$A \cong \prod_{i=1}^{r} A/\mathfrak{m}_i^n.$$

In particular, A is a finite direct product of local rings.

Proof. Let N be the nilradical of A. Since A is noetherian (it is a finitely generated module over a noetherian ring), then N is finitely generated, say $N = \langle y_1, \ldots, y_s \rangle$. For each $j = 1, \ldots, s$, there is a smallest integer $n_i > 1$ such that $y_i^{n_i} = 0$. If we set

$$n := 1 + \sum_{i=1}^{s} (n_i - 1),$$

then $N^n = 0$. Indeed, an element of N^n is a linear combination of monomials of the form $y_1^{m_1} \cdots y_s^{m_s}$ with $\sum m_i = n$, and by definition of n, at least one m_i satisfies $m_i \geq n_i$ so that the monomial is zero. Notice that n is the minimal exponent that does annihilates N.

By Lemma 1, we then have:

$$\prod \mathfrak{m}_i^n \subseteq \bigcap \mathfrak{m}_i^n = \left(\bigcap \mathfrak{m}_i\right)^n = N^n = 0.$$

Now, observe that for $i \neq j$, $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are coprime. So

$$\sqrt{\mathfrak{m}_i^n + \mathfrak{m}_j^n} = \sqrt{\sqrt{\mathfrak{m}_i^n} + \sqrt{\mathfrak{m}_j^n}} = \sqrt{\mathfrak{m}_i + \mathfrak{m}_j} = \sqrt{(1)}$$

which implies that $\mathfrak{m}_i^n + \mathfrak{m}_j^n$, i.e. the factors of $\prod \mathfrak{m}_i^n$ are pairwise comprime and hence $\cap \mathfrak{m}_i^n = \prod \mathfrak{m}_i = 0$. Therefore the map $A \to \prod A/\mathfrak{m}_i^n$ is an isomorphism. $\square$

Remark 3. Corollary 2 is half of the structure theorem for Artin rings that states that an Artin ring is uniquely (up to isomorphism) a finite direct product of Artin local rings.

Corollary 4. *Let A be a finite k-algebra. If A is diagonalizable, then A is reduced.*

Proof. Since A is diagonalizable, then $A \cong k^n$ as k-algebras; let us write $\psi : A \to k^n$ for this isomorphism. For $i = 1, \ldots, n$, we have the projection maps $k^n \to k$ which when precomposed with ψ give us surjective k-algebra homomorphisms $\pi_i : A \to k$. In fact, $\psi(x) = (\pi_1(x), \ldots, \pi_n(x))$. Furthermore, $\ker \pi_i$ is a maximal ideal for all i. Thus, Lemma 1 implies that

$$N \subseteq \bigcap_{i=1}^{n} \ker \pi_i,$$

where N is the nilradical of A. Finally, if $x \in \ker \pi_1 \cap \cdots \cap \ker \pi_n$, then $\psi(x) = 0$ and thus $x = 0$ since ψ is an isomorphism. We conclude that $N = 0$ as required. $\square$

We now have the tools to characterize finite k-algebras.

Proposition 5. *Let A be a finitely generated k-algebra. Then the following are equivalent:*

1. *A is a finite k-algebra.*
2. *A is artinian.*
3. $\dim A = 0$.
4. $\operatorname{Specm}(A)$ *is a finite discrete space.*

Proof. We prove the following implications.

($1 \implies 2$) Since k is artinian and A is a finitely generated k-module, then A is artinian.

($2 \implies 3$) Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and let $x \in A/\mathfrak{p}$ nonzero. Since A is artinian, then so is $A/\mathfrak{p}$ by the correspondence theorem. Thus the descending chain $(x) \supseteq (x^2) \supseteq \cdots$ stabilizes, that is, $(x^n) = (x^{n+1})$ for some $n > 1$. In other words, $x^n = x^{n+1}y$ for some $y \in A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is an integral domain, we may cancel x^n to get $1 = xy$, i.e. x is a unit and hence $A/\mathfrak{p}$ is a field. This shows that every prime ideal of A is maximal, i.e. $\dim A = 0$.

($3 \implies 1$) By the Noether normalization lemma, there exists algebraically independent elements $x_1, \ldots, x_r \in A$ such that A is finite over $k[x_1, \ldots, x_r]$. Since $\dim A = 0$, then we must have $r = 0$. Hence A is finite over k.

($1 \implies 4$) Let $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ be the set of maximal ideals of A. By Corollary 2, we have $A \cong \prod A_i$ where each A_i is a local ring with maximal ideal $\mathfrak{m}_i$, i.e. $\mathrm{Specm}(A_i) = \{\mathfrak{m}_i\}$. Hence

$$\mathrm{Specm}(A) = \bigsqcup \mathrm{Specm}(A/\mathfrak{m}_i) = \bigsqcup \{\mathfrak{m}_i\}.$$

Thus $\mathrm{Specm}(A)$ is discrete.

($4 \implies 3$) Let $\mathfrak{p} \subset A$ be a prime ideal and suppose $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{p}$. By assumption $\{\mathfrak{m}\} \subset \mathrm{Specm}(A)$ is open. Then there exists $f \in A$ such that $\mathrm{Specm}(A_f) = \{\mathfrak{m}\}$. Since A_f is a finitely generated k-algebra, the Nullstellensatz implies (with a similar argument as in the proof of Lemma 1) that every prime ideal $\mathfrak{q} \subset A_f$ is an intersection of maximal ideals. But A_f has exactly one of them and hence $\mathfrak{q}$ is maximal. Thus $\dim A_f = 0$

□

We can know characterize the notion of étale k-algebras.

Theorem 6. *Let A be a finite k-algebra. Then the following are equivalent:*

1. *A is étale.*
2. *$K \otimes_k A$ is reduced for every algebraic extension K of k.*
3. *$A \cong K_1 \times \cdots \times K_r$ as k-algebras, where K_i is a finite separable extension of k.*
4. *$\overline{k} \otimes_k A$ is a diagonalizable $\overline{k}$-algebra.*

Proof. We prove the following cycle of implications.

($1 \implies 2$) Let K be a field containing k. By definition, there is a field L containing k such that $L \otimes A$ is a diagonalizable L-algebra. Next, take a field F that contains both L and K (for example, one may take the quotient of $L \otimes_k K$ by some maximal ideal). Then

$$F \otimes_k A \cong (F \otimes_L L) \otimes_k A \cong F \otimes_L (L \otimes_k A) \cong F \otimes_L L^n \cong F^n.$$

That is, $F \otimes_k A$ is a diagonalizable F-algebra; by Corollary 4, $F \otimes_k A$ is reduced. Since A is a free k-module, then it is flat. Thus the inclusion map $K \hookrightarrow F$ remains injective after tensoring with A. Thus we may view $K \otimes_k A$ as a subring of $F \otimes_k A$. Thus $K \otimes_k A$ is also reduced.

($2 \implies 3$) By assumption $A \cong k \otimes_k A$ is reduced, so its nilradical is zero so Lemma 1, tells us that $A \cong K_1 \times \cdots \times K_r$ where K_i is a field extension of k. If one of the factors is an infinite field extension, then A would be infinite dimensional, so every factor K_i is a finite extension of k.

Now, suppose that K_1 is not separable. Then necessarily, $\mathrm{char}(k) = p > 0$ and there is an element $\alpha \in K_1$ whose minimal polynomial is of the form $g(x^p)$ for some $g \in k[x]$. If we write $g(x) = g_0 + \cdots + g_{n-1}x^{n-1} + x^n$, then over the field

$$L = K(\sqrt[p]{g_0}, \dots, \sqrt[p]{g_{n-1}})$$

we have that $g(x^p) = h(x)^p$ for some $h(x) \in L[x]$; in fact $h(x) = \sum h_i x^i$ with $h_i = \sqrt[p]{g_i}$. This implies that

$$L \otimes_k k[\alpha] \cong L \otimes_k \frac{k[x]}{(g(x^p))} \cong \frac{L[x]}{(g(x^p))} \cong \frac{L[x]}{(h(x))^p},$$

which is clearly not reduced. However, $L \otimes_k k[\alpha] \hookrightarrow L \otimes_k K_1 \subset L \otimes_k A$, which would imply that $L \otimes_k A$ is not reduced. However, this contradicts our assumption.

($3 \implies 4$) Suppose $A \cong K_1 \times \cdots \times K_r$ with K_i a finite separable extension of k. By the primitive root theorem, $K_i = k[\alpha_i]$ for some $\alpha_i \in \overline{k}$. The minimal polynomial $f_i \in k[x]$ of α_i is separable, so it splits completely over $\overline{k}$ into distinct linear factors. Thus

$$\overline{k} \otimes_k K_i \cong \overline{k} \otimes_k k[\alpha_i] \cong \overline{k} \otimes_k \frac{k[x]}{(f_i)} = \frac{\overline{k}[x]}{(f_i)} \cong \overline{k} \times \cdots \times \overline{k}.$$

Thus

$$\overline{k} \otimes_k A \cong \overline{k} \otimes_k (K_1 \times \cdots \times K_r) \cong (\overline{k} \otimes_k K_1) \times \cdots \times (\overline{k} \otimes_k K_r) \cong (\overline{k} \times \cdots \times \overline{k}) \times \cdots \times (\overline{k} \times \cdots \times \overline{k}).$$

Therefore $\overline{k} \otimes_k A$ is diagonalizable.

($4 \implies 1$) By definition.

□

2 Galois theory of étale algebras

Let k be a field and fix a separable closure k^{sep} and let $G = \text{Gal}(k^{\text{sep}}/k)$. A set X is a *G-set* if there is a group action $G \times X \to X$ that is continuous with respect to the Krull topology on G and the discrete topology on X. By definition of the Krull topology, the action is continuous if and only if for every $x \in X$, the stabilizer subgroup G_x is open in G.

Let A be an étale algebra over k. Then G acts on $\text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$ as follows:

$$G \curvearrowright \text{Hom}_{k\text{-alg}}(A, k^{\text{sep}}) \quad \text{with} \quad (\sigma, f) \mapsto \sigma f \quad \text{defined by} \quad (\sigma f)(a) := \sigma(f(a)). \tag{1}$$

So we a G action on $\text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$. Conversely, let X be a finite G-set. Then $\text{Hom}_{G\text{-set}}(X, k^{\text{sep}})$ is a k-algebra where the operations are defined pointwise. Furthermore, G acts on this k-algebra as follows:

$$G \curvearrowright \text{Hom}_{G\text{-set}}(X, k^{\text{sep}}) \quad \text{with} \quad (\sigma, f) \mapsto \sigma f \quad \text{defined by} \quad (\sigma f)(x) = \sigma(f(\sigma^{-1}x)).$$

Thus we can form the fixed points $\text{Hom}_{G\text{-set}}(X, k^{\text{sep}})^G$ of this action. Notice that

$$f \in \text{Hom}_{G\text{-set}}(X, k^{\text{sep}})^G \quad \iff \quad f(\sigma x) = \sigma(f(x)) \quad \forall x \in X, \ \sigma \in G.$$

It is clear that $\text{Hom}_{G\text{-set}}(X, k^{\text{sep}})^G$ is a k-subalgebra of $\text{Hom}_{G\text{-set}}(X, k^{\text{sep}})$.

Above, we have defined functors between the category of finite G-sets and the category of étale algebras. We make this statement precise and prove it below.

Theorem 7. *Let k be a field, k^{sep} a choice of separable closure and $G = \text{Gal}(k^{\text{sep}}/k)$ the absolute Galois group. Let $\textbf{Ét-Alg}_k$ be the category of étale k-algebras and $\textbf{FinSet}_G$ be the category of finite G-sets. Then there are functors*

$$\mathcal{F} : \textbf{Ét-Alg}_k \longrightarrow \textbf{FinSet}_G \qquad A \mapsto \text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$$
$$\mathcal{G} : \textbf{FinSet}_G \longrightarrow \textbf{Ét-Alg}_k \qquad X \mapsto \text{Hom}_{G\text{-set}}(X, k^{\text{sep}})^G$$

These functors define an equivalence of categories.

Proof. We break up the proof into several steps.

1. ($\mathcal{F}$ is well-defined) First we show that the action of G on $\text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$, defined in (1) is continuous. To see this, we write $A \cong K_1 \times \cdots \times K_r$ where K_i is a finite separable extension of k. Then we embed $K_i \hookrightarrow k^{\text{sep}}$ and take the Galois closure L of the compositum $K_1 \cdots K_r$. This is a finite Galois extension since each K_i is finite. This means that for any $f \in \text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$, the image $f(A)$ lies in L, so $\text{Gal}(k^{\text{sep}}/L) \subseteq G_f$. Since $\text{Gal}(k^{\text{sep}}/L)$ is open of finite index, then so is G_f (it is the finite union of translates of $\text{Gal}(k^{\text{sep}}/L)$) and hence the above action is continuous.

   Next, we show that $\text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$ is finite. If $A = A_1 \times \cdots \times A_s$ is a product of étale algebras (which is again étale by Theorem 6), then any map $f \in \text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$ must be zero on all factors except (possibly) one factor. Indeed, k^{sep} is an integral domain. Thus we have

   $$\text{Hom}_{k\text{-alg}}\left(\prod A_i, k^{\text{sep}}\right) = \bigsqcup \text{Hom}_{k\text{-alg}}(A_i, k^{\text{sep}}).$$

   In particular, if A is an étale algebra with $A \cong \prod K_i$, then

   $$\text{Hom}_{k\text{-alg}}(A, k^{\text{sep}}) = \bigsqcup \text{Hom}_{k\text{-alg}}(K_i, k^{\text{sep}}). \tag{2}$$

   Hence $\text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$ is finite of order $\dim_k A$ since each K_i is separable.

   Finally, we must show that the k-algebra homomorphisms get mapped to G-equivariant maps under $\mathcal{F}$. If $f : A \to B$ is a k-algebra homomorphism, then $\mathcal{F}(f) : \text{Hom}_{k\text{-alg}}(B, k^{\text{sep}}) \to \text{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$ is simply precomposition by f. Thus, if $\phi : B \to k^{\text{sep}}$ is a k-algebra homomorphism, $\sigma \in G$ and $b \in B$, then

   $$\mathcal{F}(f)(\sigma\phi)(b) = (\sigma\phi \circ f)(b) = (\sigma\phi)(f(b)) = \sigma(\phi(f(b))) = \sigma(\mathcal{F}(f)(\phi)(b)) = \sigma\mathcal{F}(f)(\phi)(b).$$

   That is, $\mathcal{F}(f)$ is G-equivariant as required.

2. ($\mathcal{G}$ is well-defined) Notice that $\mathcal{G}$ is just the composition of the functors $\text{Hom}_{G\text{-set}}(-, k^{\text{sep}})$ and the fixed point functor so functoriality is automatic; we need only prove that $\mathcal{G}$ has the correct domain, i.e. $\text{Hom}_{G\text{-set}}(X, k^{\text{sep}})^G$ is indeed an étale algebra.

Suppose that X is a finite G-set. Then we can decompose X into a disjoint union of orbits, say $X = X_1 \sqcup \cdots \sqcup X_r$ with some choice of representative $x_i \in X_i$. There is a natural bijection

$$\mathrm{Hom}_{G\text{-set}}(X, k^{\mathrm{sep}}) = \prod \mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}}).$$

The fixed point functor is left exact so it preserves finite limits (i.e. products) so we have

$$\mathrm{Hom}_{G\text{-set}}(X, k^{\mathrm{sep}})^G = \prod \mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}})^G. \tag{3}$$

If $f \in \mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}})^G$ and $x \in X_i$ is arbitrary, then $x = \sigma x_i$ for some $\sigma \in G$ so that $f(x) = f(\sigma x_i) = f(x_i)$. This means that a map in $\mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}})^G$ is completely determined by its value on x_i. Furthermore, if $\sigma \in G_{x_i}$, then $\sigma x_i = x_i$ and thus $\sigma f(x_i) = f(x_i)$, which means that $f(x_i)$ lies in the fixed field $L_i := (k^{\mathrm{sep}})^{G_{x_i}}$. Since the action $G \curvearrowright X$ is continous, then G_{x_i} is open and thus L_i is a finite extension of k. Conversely, any element α of L_i makes the map $f : X_i \to k^{\mathrm{sep}}$, defined by $x_i \mapsto \alpha$, invariant under the action of G. Hence we get a k-algebra isomorphism

$$\mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}})^G \xrightarrow{\sim} L_i \quad \text{defined by} \quad f \mapsto f(x_i),$$

since the k-algebra operations are defined pointwise. Thus we get an isomorphism

$$\mathrm{Hom}_{G\text{-set}}(X, k^{\mathrm{sep}})^G = \prod_{i=1}^{r} L_i$$

and therefore $\mathrm{Hom}_{G\text{-set}}(X, k^{\mathrm{sep}})^G$ is an étale algebra by Theorem 6.

3. ($\mathcal{F}$ is fully faithful) Let A and B be étale k-algebras. Then $\mathcal{F}$ defines a map

$$\mathrm{Hom}_{k\text{-alg}}(A, B) \longrightarrow \mathrm{Hom}_{G\text{-set}}(\mathrm{Hom}_{k\text{-alg}}(B, k^{\mathrm{sep}}), \mathrm{Hom}_{k\text{-alg}}(A, k^{\mathrm{sep}}))$$

with $f : A \to B$ mapping to the function $f^* : \phi \mapsto \phi \circ f$. It is well-known that

$$\mathrm{Hom}_k(A, B) \cong \mathrm{Hom}_{k^{\mathrm{sep}}}(k^{\mathrm{sep}} \otimes_k A, k^{\mathrm{sep}} \otimes_k B)^G,$$

where the Hom sets are k-linear and k^{sep}-linear maps respectively. Since k-algebra homomorphisms on the LHS correspond to k^{sep}-algebra homomorphisms on the RHS, then we obtain

$$\mathrm{Hom}_{k\text{-alg}}(A, B) \cong \mathrm{Hom}_{k^{\mathrm{sep}}\text{-alg}}(k^{\mathrm{sep}} \otimes_k A, k^{\mathrm{sep}} \otimes_k B)^G.$$

Since A (resp. B) is étale, then $k^{\mathrm{sep}} \otimes_k A$ (resp. $k^{\mathrm{sep}} \otimes_k B$) is diagonalizable and hence a finite product of k^{sep}'s. Then by (2) we see that...

$$\vdots$$

4. ($\mathcal{F}$ is essentially surjective) Let X be a finite G-set. Consider

$$\epsilon : X \longrightarrow \mathcal{F}\mathcal{G}(X) = \mathrm{Hom}_{k\text{-alg}}(\mathrm{Hom}_{G\text{-set}}(X, k^{\mathrm{sep}})^G, k^{\mathrm{sep}}) \quad \text{with} \quad x \mapsto \epsilon_x$$

where ϵ_x is the evaluation at x map:

$$\epsilon_x : \mathrm{Hom}_{G\text{-set}}(X, k^{\mathrm{sep}})^G \longrightarrow k^{\mathrm{sep}} \quad \text{is defined by} \quad \epsilon_x(g) = g(x).$$

Below we show that ϵ is a G-equivariant set bijection.

Firts observe that if $X = X_1 \sqcup \cdots \sqcup X_r$ is the decomposition into orbits, then by (3) we have

$$\begin{aligned}
\mathcal{F}\mathcal{G}(X) &= \mathrm{Hom}_{k\text{-alg}}(\mathrm{Hom}_{G\text{-set}}(X, k^{\mathrm{sep}})^G, k^{\mathrm{sep}}) \\
&\cong \mathrm{Hom}_{k\text{-alg}}\left(\prod \mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}})^G, k^{\mathrm{sep}}\right) \\
&\cong \bigsqcup \mathrm{Hom}_{k\text{-alg}}(\mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}})^G, k^{\mathrm{sep}}).
\end{aligned}$$

Thus it is enough to show that $X_i \cong \mathrm{Hom}_{k\text{-alg}}(\mathrm{Hom}_{G\text{-set}}(X_i, k^{\mathrm{sep}})^G, k^{\mathrm{sep}})$. In other words, it is enough to show that ε is bijective when the action $G \curvearrowright X$ is transitive so we make this assumption going forward.

$$\vdots$$

□

3 The module of Kähler differentials

We can also characterize étale algebras by the vanishing of its module of Kähler differentials.

Definition 2. Let A be a k-algebra (k not necessarily a field) and M and A-module. A *k-derivation* of A with values in M, is a k-linear map $d : A \to M$ that satisfies the *Leibniz rule*:

$$d(xy) = xd(y) + yd(x).$$

The set of such derivations is denoted by $\mathrm{Der}_k(A, M)$.

Theorem 8. *Let A be a k-algebra (k not necessarily a field). There exists an A-module $\Omega^1_{A/k}$ and a k-derivation $d_{A/k} : A \to \Omega^1_{A/k}$ that satisfy the following universal property: for any A-module M and any k-derivation $d \in \mathrm{Der}_k(A, M)$, there exists a unique A-module morphism $\phi : \Omega^1_{A/k} \to M$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\quad d \quad} & M \\ {\scriptstyle d_{A/k}}\downarrow & \nearrow{\scriptstyle \phi} & \\ \Omega^1_{A/k} & & \end{array} \tag{4}$$

Remark 9. The universal property is equivalent to saying that there is a natural bijection

$$\mathrm{Hom}_A(\Omega^1_{A/k}, M) \xrightarrow{\sim} \mathrm{Der}_k(A, M) \quad \text{defined by} \quad \psi \mapsto \psi \circ d_{A/k}.$$

In fact, it is equivalent to saying that the functor

$$\mathrm{Der}_k(A, -) : {}_A\mathbf{Mod} \longrightarrow {}_A\mathbf{Mod} \quad \text{defined by} \quad M \mapsto \mathrm{Der}_k(A, M)$$

is representable and it is represented by $\Omega^1_{A/k}$. Hence $\Omega^1_{A/k}$ is unique up to isomorphism; we call it the *module of Kähler differentials* of A over k.

Proposition 10. *Let A be a k-algebra (k not necessarily a field). The module of Kähler differentials of A over k, satisfies the following properties:*

1. $\Omega^1_{A/A} = 0$
2. *If k' is a k-algebra and we set $A' := k' \otimes_k A$, then $\Omega^1_{A'/k'} \cong k' \otimes_k \Omega^1_{A/k}$.*
3. *If $S \subset A$ is a multiplicatively closed set, then $S^{-1}(\Omega^1_{A/k}) \cong \Omega^1_{S^{-1}A/k}$.*
4. *If B is another k-algebra, then $\Omega^1_{A\times B/k} \cong \Omega^1_{A/k} \times \Omega^1_{B/k}$.*
5. *If $A \to B$ is a k-algebra homomorphism, then we have the exact sequence*
$$\Omega^1_{A/k} \otimes_A B \longrightarrow \Omega^1_{B/k} \longrightarrow \Omega^1_{B/A} \longrightarrow 0.$$
6. *If $A \to B$ is a surjective k-algebra homomorphism with kernel $I \subset A$, then we can the above exact sequence to the left as*
$$I/I^2 \longrightarrow \Omega^1_{A/k} \otimes_A B \longrightarrow \Omega^1_{B/k} \longrightarrow 0.$$
7. *If $A \to k$ is a surjective k-algebra homomorphism with kernel $I \subset A$, then we have a canonical isomorphism*
$$\Omega^1_{A/k} \cong I/I^2.$$

We can now state and prove the characterization of étale algebras in terms of their Kähler differentials.

Theorem 11. *Let A be a finite k-algebra. Then A is étale if and only if $\Omega^1_{A/k} = 0$.*

Proof. Set $\overline{A} := \overline{k} \otimes_k A$.

($\Longrightarrow$) By Theorem 6, we know that $\overline{A} \cong \overline{k} \times \cdots \times \overline{k}$. Hence

$$\overline{k} \otimes_k \Omega^1_{A/k} \cong \Omega^1_{\bar{A}/\bar{k}} \cong \Omega^1_{\bar{k}/\bar{k}} \times \cdots \times \Omega^1_{\bar{k}/\bar{k}} = 0.$$

Hence $\Omega^1_{A/k} = 0$.

($\Longleftarrow$) Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of A. By Corollary 2, we can write $A = \prod A_i$ where $A_i = A/\mathfrak{m}_i^n$ are finite k-algebras that are local. By assumption

$$0 = \Omega^1_{A/k} \cong \prod \Omega^1_{A_i/k},$$

so $\Omega^1_{A_i/k} = 0$ for all i. Since $\overline{k} = k$, then $A_i/\mathfrak{m}_i = k$. Thus the projection map $A_i \to A_i/\mathfrak{m}_i = k$ tells us that $\Omega^1_{A_i/k} \cong \mathfrak{m}_i/\mathfrak{m}_i^2$ canonically. Hence $\mathfrak{m}_i = \mathfrak{m}_i^2$ so that Nakayama's Lemma implies that $\mathfrak{m}_i = 0$. That is, $A_i \cong k$ so that A is diagonalizable, hence étale.

For the case when $k \neq \bar{k}$, the above argument says that $\bar{A}$ is étale, i.e. reduced. For any algebraic extension K of k, we have $K \otimes_k A \hookrightarrow \bar{A}$ so that $K \otimes_k A$ is reduced. Thus A is étale.

□

4 Étale Group Schemes

Let $X = \mathrm{Spec}A$ be an affine scheme over k. We say the X is *étale* if A is an étale k-algebra. Then we have the following scheme-theoretic version of Theorem 6.

Theorem 12. *Let $X = \mathrm{Spec}A$ be an affine scheme finite over k, i.e. A is a finite k-algebra. Then the following are equivalent:*

1. *X is étale.*
2. *X is geometrically reduced.*
3. *X is smooth.*

Proof. If A is an étale, then $\bar{k} \otimes_k A$ is reduced, i.e. X is geometrically reduced. If X is geometrically reduced, then $\bar{k} \otimes A$ is reduced and thus $K \otimes_k A \hookrightarrow \bar{k} \otimes_K A$ is as well for any algebraic extension K of k. Hence A is étale.

For $\mathfrak{p} \in \mathrm{Spec}(A)$, we have $T_\mathfrak{p}X = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$. Since $A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{m}_p = k$ is surjective, then

$$(T_\mathfrak{p}X)^\vee \cong \mathfrak{m}_p/\mathfrak{m}_p^2 \cong \Omega^1_{A_p/k} \cong (\Omega^1_{A/k})_\mathfrak{p}.$$

Thus

$$\dim_k T_\mathfrak{p}X = 0 \quad \forall \mathfrak{p} \quad \Longleftrightarrow \quad (\Omega^1_{A/k})_\mathfrak{p} = 0 \quad \forall \mathfrak{p} \quad \Longleftrightarrow \quad \Omega^1_{A/k} = 0.$$

Thus if X is smooth, then $\dim_k T_\mathfrak{p}X = \dim X$

□

Now let us work with group schemes. Let $\mathcal{G}$ be a finite group scheme over k, so $G = \mathrm{Spec}A$ for a Hopf algebra A which is a finite k-algebra. In this case, A is finite dimensional over k, so we define the *order* of $\mathcal{G}$ to be

$$o(\mathcal{G}) := \dim_k(A).$$

The order of a finite group scheme determines its smoothness.

Theorem 13. *Let $\mathcal{G}$ be a finite group scheme over k. If $o(G)$ is invertible in k, then G is étale.*

Proof. We separate into two cases:

1. ($\mathrm{char}(k) = 0$) In this case, Cartier's theorem states that $\mathcal{G}$ is smooth hence étale by Theorem 12.
2. ($\mathrm{char}(k) = p$) Consider the following map:

$$\phi_r : A \longrightarrow A \quad \text{defined by} \quad a \mapsto a^{p^r}.$$

This map is the p^rth Frobenius map and it is a k-algebra homomorphism since $\mathrm{char}(k) = p$. Since A is noetherian, then its nilradical N is annihilated by some exponent $n > 0$ (see the proof of corollary (2)). So if r is such that $p^r > n$, then $\phi_r(A) = A^{p^r}$ is reduced. Thus for sufficiently large r, the image of the Frobenius map is smooth.

□

The galois theory of étale algebras translates to the theory of group schemes as follows.

Theorem 14. *The functor $X \mapsto X(k^{\mathrm{sep}})$ is an equivalence between the categories of étale schemes over k and the category of finite discrete G-sets. Moreover, the functor $\mathcal{G} \mapsto \mathcal{G}(k^{\mathrm{sep}})$ is an equivalence between the category of finite étale group schemes and finite (discrete) groups with a continuous action of* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.