Brauer Groups of Schemes

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Chapter 1

Azumaya Algebras

Throughout this chapter we fix the following notation:

- R is a commutative Noetherian local ring with (unique) maximal ideal $\mathfrak{m}$ and residue field by $k := R/\mathfrak{m}$.
- A is a unital ring, not necessarily commutative, with the structure of an R-algebra.
- $X = (X, \mathcal{O}_X)$ is a locally noetherian scheme. That is, there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $U_i \cong \mathrm{Spec} A_i$, as schemes, where each A_i is a noetherian ring.
- $\mathcal{A}$ is an $\mathcal{O}_X$-algebra, that is, it is a sheaf of abelian groups such that for each open subset $U \subseteq X$, the abelian group $\mathcal{A}(U)$ is an $\mathcal{O}_X(U)$-algebra whose structure homomorphism is compatible with the sheaf structure of $\mathcal{A}$ and $\mathcal{O}_X$.

In this chapter we describe Azumaya algebras. These can be defined over local rings and then generalized to schemes through the local version.

1.1 Azumaya Algebras over Local Rings

The definition of Azumaya algebras over a local base ring R is slightly easier than when R is not local (more precisely, a module being locally free is much easier to verify when R is a noetherian local domain, cf. Proposition A.9). We will use the following definition:

Definition. We say that A is an *Azumaya algebra over R* if

(i) A is a free R-module of finite rank,

(ii) The map

$$\Psi_{A/R} : A \otimes_R A^{\mathrm{op}} \longrightarrow \mathrm{End}_{R\text{-mod}}(A) \quad \text{defined by} \quad a \otimes a' \mapsto (x \mapsto axa') \tag{1.1}$$

is an R-algebra isomorphism.

We recall some basic properties of Azumaya algebras that we will require later.

Proposition 1.1. *Let A be an R algebra.*

(a) Suppose $R = K$ is a field, then:

$$A \text{ is an Azumaya algebra over } K \quad \Longleftrightarrow \quad A \text{ is a finite dimensional, central simple algebra over } K.$$

(b) Let R' be a local (commutative) R-algebra. Then

$$A \text{ is an Azumaya algebra over } R \quad \Longrightarrow \quad A \otimes_R R' \text{ is an Azumaya algebra over } R'.$$

(c) Suppose that A is a free R-module of finite rank, then

$$A \otimes_R R/\mathfrak{m} \text{ is an Azumaya algebra over } R/\mathfrak{m} \quad \Longrightarrow \quad A \text{ is an Azumaya algebra over } R.$$

Proof.

(a) Being a free K-module of finite rank is equivalent to being a finite dimensional K-vector space, so the content of this proposition is showing that condition (ii) of being an Azumaya algebra over K is equivalent to being central and simple.

($\Longrightarrow$) Let A be Azumaya over K. First we extend $\{1\} \subset A$ to a basis $\{1 = a_1, \dots, a_n\}$ of A over K and let $\chi_1, \dots, \chi_n \in \mathrm{End}_K(A)$ be defined on the basis elements as

$$\chi_i(a_j) = \delta_{ij} = \text{the Kronecker delta function.}$$

Next, let $z \in Z(A)$, which can be written in the form $z = r_1 a_1 + \cdots + r_n a_n$ with $r_i \in K$. Then

$$z = 1 \cdot z = \chi_1(1) \cdot z. \tag{1.2}$$

Since A is Azumaya over K, then the map $\Psi_{A/K}$ defined in (1.1) is a K-algebra isomorphism. Thus

$$\exists \sum_{i=1}^{m} b_i \otimes c_i \in A \otimes A^{\mathrm{op}} \quad \text{such that} \quad \Psi_{A/K}\left(\sum_{i=1}^{m} b_i \otimes c_i\right) = \chi_1$$

$$\therefore \qquad \chi_1(a) = \sum_{i=1}^{m} b_i a c_i \qquad \forall a \in A.$$

Thus, since $z \in Z(A)$, we have

$$\chi_1(az) = \sum_{i=1}^{m} b_i a z c_i = \sum_{i=1}^{m} b_i a c_i z = \chi_1(a) z \qquad \forall x \in A.$$

Using this identity with $a = a_1 = 1$ in (1.2) gives:

$$z = \chi_1(1) z = \chi_1(1 \cdot z) = \chi_1(z) = \chi_1\left(\sum_{i=1}^{n} r_i a_i\right) = \sum_{i=1}^{n} r_i \delta_{1i} = r_1 \in K.$$

This proves that $Z(A) \subseteq K$. The other inclusion is trivial because K is commutative. We have shown that A is central.

Next, let $I \leq A$ be a two-sided ideal and $x \in I$. Observe that

$$\chi_1(x) = \sum_{i=1}^{m} b_i x c_i \in I$$

since I is two-sided. This shows that $\chi_1(I) \subseteq I$; of course this argument works for any endomorphism, that is:

$$\chi(I) \subseteq I \qquad \forall \chi \in \mathrm{End}_K(A), \tag{1.3}$$

since every endomorphism is "multiplication" by an element of $A \otimes A^{\mathrm{op}}$. Now, if we write $x = \sum r_i a_i$, with $r_i \in K$, then each coefficient satisfies:

$$r_j = \sum_{i=1}^{n} r_i \delta_{ij} = \chi_j\left(\sum_{i=1}^{n} r_i a_i\right) = \chi_j(x).$$

The LHS is an element of K while the RHS is an element of I by (1.3). Therefore $r_j \in K \cap I$ which is an ideal of K. Since K is a field, then $I \cap K = 0$ or $I \cap K = K$. In the first case, we have that $r_1 = \cdots = r_n = 0$ and so $x = 0$; that is $I = 0$. In the second case, we would have $1 \in K = I \cap K \subseteq I$ and hence $I = A$. This proves that A is simple.

($\Longleftarrow$) A well-known property of central simple algebras states that if A is central simple, then both A^{op} and $A \otimes A^{\mathrm{op}}$ are central simple K-algebras (any text on central simple algebras over a field will contain this result, see for example Proposition 2.36 of Chapter II of [Kna07]). This implies that the K-algebra (i.e. K-linear) map

$$\Psi_{A/K} : A \otimes_K A^{\mathrm{op}} \longrightarrow \mathrm{End}_K(A)$$

has trivial kernel, since the kernel is always a two-sided ideal and the map is nonzero since $1 \otimes 1$ maps to the identity map. Finally, the map is an isomorphism because of the following dimension count:

$$\dim_K(A \otimes A^{\mathrm{op}}) = \dim_K(A)^2 = \dim_K\big(\mathrm{Mat}_{n\times n}(A)\big) = \dim_K\big(\mathrm{End}_K(A)\big)$$

where these dimensions are finite. This shows that A is an Azumaya algebra over K.

(*b*) First, we tensor the map $\Psi_{A/R}$ with $\mathrm{id}_{R'}$ to get

$$(\Psi_{A/R} \otimes \mathrm{id}_{R'}) : (A \otimes_R A^{\mathrm{op}}) \otimes_R R' \longrightarrow \mathrm{End}_R(A) \otimes_R R'.$$

We also have the map

$$\Psi_{A\otimes R'/R'} : (A \otimes_R R') \otimes_R' (A \otimes_R R')^{\mathrm{op}} \longrightarrow \mathrm{End}_{R'}(A \otimes_R R')$$

We will see that they fit in the following diagram:

$$\begin{array}{ccc} (A \otimes_R A^{\mathrm{op}}) \otimes_R R' & \xrightarrow{\Psi_{A/R}\otimes \mathrm{id}_{R'}} & \mathrm{End}_R(A) \otimes_R R' \\ \downarrow \alpha & & \downarrow \beta \\ (A \otimes_R R') \otimes_{R'} (A \otimes_R R')^{\mathrm{op}} & \xrightarrow{\Psi_{A\otimes R'/R'}} & \mathrm{End}_{R'}(A \otimes_R R') \end{array} \tag{1.4}$$

The left vertical arrow is a canonical isomorphism obtained by

$$(A \otimes_R A^{\mathrm{op}}) \otimes_R R' \cong A \otimes_R (A^{\mathrm{op}} \otimes_R R') \cong A \otimes_R \left(R' \otimes_{R'} (A^{\mathrm{op}} \otimes_R R')\right) \cong (A \otimes_R R') \otimes_{R'} (A \otimes_R R')^{\mathrm{op}}$$

The right vertical arrow is the R'-algebra homomorphism

$$\beta : \mathrm{End}_R(A) \otimes_R R' \longrightarrow \mathrm{End}_{R'}(A \otimes_R R') \quad \text{defined by} \quad \varphi \otimes 1_{R'} \mapsto \varphi \otimes \mathrm{id}_{R'}$$

and this map is an isomorphism whenever A is a free R-module by Proposition A.7, which occurs in our case since A is Azumaya over R by assumption.

The diagram (1.4) commutes. We can check this on the simple tensors of the form $(a \otimes a') \otimes r' \in (A \otimes_R A^{\mathrm{op}}) \otimes_R R'$ since these generate the whole algebra. A straightforward computation, using the formula for β from Proposition A.7 yields

$$\begin{array}{ccc} (a \otimes a') \otimes r' & \longmapsto & (x \mapsto axa') \otimes r' \\ \downarrow & & \downarrow \\ r'(a \otimes 1) \otimes (a' \otimes 1) & \longmapsto & \left((x \otimes r) \mapsto r'(a \otimes 1)(x \otimes r)(a' \otimes 1) = r'(axa' \otimes r)\right) \end{array}$$

Finally, since A is Azumaya over R, then $\Psi_{A/R}$ is an isomorphism by assumption so the commutativity of the diagram implies that $\Psi_{A/R} \otimes \mathrm{id}_{R'}$ is an isomorphism as well.

(*c*) By setting $R' = R/\mathfrak{m} = k$ in (1.4) we obtain

$$\begin{array}{ccc} (A \otimes_R A^{\mathrm{op}}) \otimes_R k & \xrightarrow{\Psi_{A/R}\otimes \mathrm{id}_{k}} & \mathrm{End}_R(A) \otimes_R k \\ \downarrow \alpha & & \downarrow \beta \\ (A \otimes_R k) \otimes_{k} (A \otimes_R k)^{\mathrm{op}} & \xrightarrow{\Psi_{A\otimes k/k}} & \mathrm{End}_{k}(A \otimes_R k) \end{array} \tag{1.5}$$

By assumption, the bottom horizontal arrow is an isomorphism so the top horizontal arrow is an isomorphism as well. Furthermore, A is a free R-module of finite rank so that $\mathrm{End}_R(A)$ is also a free R-module of finite rank. This means we can apply Proposition A.5 an conclude that $\Psi_{A/R}$ is an isomorphism.

□

1.2 Azumaya Algebras over Schemes

We begin by extending the definition of Azumaya Algebra to schemes in the same way most commutative algebra constructions are extended to schemes.

Definition 1. Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. We say that $\mathcal{A}$ is an *Azumaya Algebra over X* if

(1.i) $\mathcal{A}$ is coherent as an $\mathcal{O}_X$-module,

(1.ii) For every closed point x of X, the stalk $\mathcal{A}_x$ is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$.

Remarks 1.2. (about Azumaya algebras over a scheme)

(1.2.1) We may drop the requirement that $x \in X$ be a closed point in condition (1.ii). That is:

$$\mathcal{A}_x \text{ is Azumaya over } \mathcal{O}_{X,x} \text{ for all } \textbf{closed} \text{ points } x \in X \quad \Longleftrightarrow \quad \mathcal{A}_x \text{ is Azumaya over } \mathcal{O}_{X,x} \ \ \forall x \in X.$$

Proof. Let $x \in X$ be any point, and choose an open affine neighborhood $U = \mathrm{Spec} B$ of x for which $\mathcal{A}|_U \cong \widetilde{M}$ for some B-module M. If x corresponds to a prime ideal $\mathfrak{p} \subset B$, then $\mathcal{A}_x = M_{\mathfrak{p}}$. Next, choose a maximal ideal $\mathfrak{q}$ containing $\mathfrak{p}$ so that the localization map $B_{\mathfrak{q}} \to (B_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}} \cong B_{\mathfrak{p}}$ (cf. Proposition A.1)makes $B_{\mathfrak{p}}$ into a $B_{\mathfrak{q}}$-algebra. If y is the closed point of X that corresponds to $\mathfrak{q}$, then condition (1.ii) tells us that $\mathcal{A}_y = M_{\mathfrak{q}}$ is an Azumaya algebra over $\mathcal{O}_{X,y} = B_{\mathfrak{q}}$. However, since

$$M_{\mathfrak{p}} \cong B_{\mathfrak{p}} \otimes_B M \cong (B_{\mathfrak{p}} \otimes_{B_{\mathfrak{q}}} B_{\mathfrak{q}}) \otimes_B M \cong B_{\mathfrak{p}} \otimes_{B_{\mathfrak{q}}} (B_{\mathfrak{q}} \otimes_B M) \cong B_{\mathfrak{p}} \otimes_{B_{\mathfrak{q}}} M_{\mathfrak{q}}$$

then $M_{\mathfrak{p}} = \mathcal{A}_x$ is Azumaya over $B_{\mathfrak{p}} = \mathcal{O}_{X,x}$ by Proposition 1.1.(c). □

(1.2.2) Any Azumaya algebra $\mathcal{A}$ over a scheme X is a locally free $\mathcal{O}_X$-module. This follows immediately from Proposition A.16.(ii).

(1.2.3) Given any Azumaya algebra $\mathcal{A}$ over X, we get a canonical $\mathcal{O}_X$-algebra homomorphism

$$\Psi : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \longrightarrow \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A}) \tag{1.6}$$

in the following manner:

For every open subset $U \subseteq X$, we can define the map

$$\psi_U : \mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{A}(U)^{\mathrm{op}} \longrightarrow \underline{\mathrm{End}}_{\mathcal{O}_{X|_U}}(\mathcal{A}|_U)$$

as follows: given an elementary tensor $s \otimes s' \in \mathcal{A}(U) \otimes \mathcal{A}(U)^{\mathrm{op}}$, we define the morphism of $\mathcal{O}_X$-algebras

$$\psi_U(s \otimes s') : \mathcal{A}|_U \longrightarrow \mathcal{A}|_U \quad \text{with} \quad \psi_U(s \otimes s')(V) : \mathcal{A}|_U(V) \to \mathcal{A}|_U(V) \quad \text{defined as} \quad x \mapsto s|_V \cdot x \cdot s'|_V$$

where $V \subseteq U$ and then we extend the definition of ψ_U to every element of $\mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{A}(U)^{\mathrm{op}}$ by linearity. It is straightforward to check that $\psi_U(s \otimes s')$ is indeed a morphism of $\mathcal{O}_X$-algebras and that the maps ψ_U define a presheaf homomorphism between the presheaf $\mathcal{G} : U \mapsto \mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{A}(U)^{\mathrm{op}}$ and the sheaf $\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})$. By passing to the sheafification of $\mathcal{G}$, we obtain the map in (1.6); it is then also straightforward to check that this map is an $\mathcal{O}_X$-algebra homomorphism.

(1.2.4) The canonical $\mathcal{O}_X$-algebra homomorphism in (1.6) induces the map (1.1) at the level of stalks. More precisely: for every $x \in X$, the map in (1.6) induces the map on stalks:

$$\Psi_x : (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}})_x \longrightarrow \left(\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})\right)_x$$

However, we have natural isomorphisms

$$(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}})_x \cong \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x^{\mathrm{op}} \quad \text{and} \quad \left(\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})\right)_x \cong \mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{A}_x)$$

by Proposition A.12, and hence we get the an $\mathcal{O}_{X,x}$-algebra homomorphism

$$\Psi_x : \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x^{\mathrm{op}} \longrightarrow \mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{A}_x) \quad \text{defined by} \quad [s] \otimes [s'] \mapsto ([y] \mapsto [s][y][s'])$$

where the formula follows from the definition of Ψ from above.

The definition of Azumaya algebra can be characterized in several ways. It is useful to relate the definition of Azumaya algebra over a scheme to the theory of Azumaya algebras over local rings and to the classical theory of central simple algebras over fields. Below, we show two other equivalent definitions of Azumaya algebras over a scheme that reflect these two theories respectively.

Theorem 1.3. *Let X be a scheme and let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent:*

1. *$\mathcal{A}$ is an Azumaya algebra over X,*
2. *$\mathcal{A}$ is a locally free $\mathcal{O}_X$-module and $\mathcal{A}_x \otimes \kappa(x)$ is a central simple algebra over the residue field $\kappa(x)$ of x, for any point x in X.*
3. *$\mathcal{A}$ is a locally free $\mathcal{O}_X$-module and the canonical homomorphism*

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \longrightarrow \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})$$

is an isomorphism.

Proof. We will prove separately that (1) $\Longleftrightarrow$ (2) and (1) $\Longleftrightarrow$ (3).

(1)$\Longrightarrow$(2) By Remark (1.2.2) we have that $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module. By assumption, for any point x in X, we have that $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$, so Proposition 1.1.(b) implies that

$$\mathcal{A}_x \otimes \kappa(x) = \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x}/\mathfrak{m}_{X,x})$$

is an Azumaya algebra over $\kappa(x)$, which is a field, and hence it is a central simple algebra over $\kappa(x)$ by Proposition 1.1.(a).

(2)$\Longrightarrow$(1) Let $x \in X$ be a closed point. By Proposition 1.1.(a), our assumption on $\mathcal{A}_x \otimes \kappa(x)$ implies that $\mathcal{A}_x \otimes \kappa(x)$ is an Azumaya algebra over $\kappa(x)$. Next, we want to apply Proposition 1.1.(c) to $\mathcal{A}_x \otimes \kappa(x)$ to conclude that $\mathcal{A}_x$ is Azumaya over $\mathcal{O}_{X,x}$. We can do this because $\mathcal{A}$ is locally free by assumption, so that $\mathcal{A}_x$ is a free $\mathcal{O}_{X,x}$-module by Proposition A.16.(ii) and has finite rank since $\mathcal{A}$ is coherent by definition. Thus we may apply Proposition 1.1.(c) and we conclude that $\mathcal{A}$ is Azumaya over X.

(1) $\Longleftrightarrow$ (3) By Remark (1.2.2) we have that $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module. So by definition, we have that $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$ for every $x \in X$ if and only if the map

$$\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x^{\mathrm{op}} \longrightarrow \mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{A}_x) \quad \text{defined by} \quad [s] \otimes [s'] \mapsto ([y] \mapsto [s][y][s'])$$

is an isomorphism. However, Remark (1.2.4) tells us that this map is the one induced on stalks from the canonical map

$$\Psi : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \longrightarrow \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})$$

from (1.6). Since scheme homomorphisms are isomorphisms if and only if they are isomorphisms at the level of sheaves, the required equivalence follows.

□

Example 1.4. Let X be a neotherian affine scheme with $X = \mathrm{Spec}R$ and let $\mathcal{A}$ be an Azumaya algebra over X. Since $\mathcal{A}$ is coherent and locally free, then $\mathcal{A}(X) = \widetilde{A}$ for some free R-module A. Since $\mathcal{A}$ is Azumaya over X, then $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \cong \underline{\mathrm{End}}(\mathcal{A})$ hence, taking global sections, we have $A \otimes_R A^{\mathrm{op}} \cong \mathrm{End}_R(A)$. Thus an Azumaya algebra $\mathcal{A}$ over the affine scheme $\mathrm{Spec}R$ is equivalent to an Azumaya algebra A over R where $\mathcal{A} = \widetilde{A}$.

1.3 The Brauer Group of a Scheme

We define the notion Brauer equivalence of Azumaya algebras over a scheme X in a similar way to how it is done for central simple algebras.

Definition 2. Let $\mathcal{A}$ and $\mathcal{A}'$ be two Azumaya algebras over a scheme X. We say that $\mathcal{A}$ is *Brauer equivalent* to $\mathcal{A}'$, denoted by $\mathcal{A} \sim \mathcal{A}'$, if there exist locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ over X such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}')$$

Remark 1.5. Of course, Brauer equivalence is an equivalence relation. Reflexivity is shown taking $\mathcal{E} = \mathcal{E}' = \mathcal{O}_X$ which is clearly a locally free $\mathcal{O}_X$-module; symmetry follows from the fact that the definition of Brauer equivalence is inherently symmetric. For transitivity, suppose $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{A}' \sim \mathcal{A}''$ via

$$\mathcal{A} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}') \quad \text{and} \quad \mathcal{A}' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}'') \cong \mathcal{A}'' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}''')$$

respectively. Now, since each $\mathcal{E}^{(i)}$ is locally free of finite rank, Proposition A.16 tells us that each stalk $\mathcal{E}_x^{(i)}$, for $x \in X$, is a free $\mathcal{O}_{X,x}$-module of finite rank. Thus Proposition A.12 applies to get

$$\mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{E}_x^{(i)} \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x^{(j)}) \cong \mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{E}_x^{(i)}) \otimes_{\mathcal{O}_{X,x}} \mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{E}_x^{(j)})$$

at the level of stalks. Hence*

$$\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}^{(i)}) \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}^{(j)}) \cong \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}^{(i)} \otimes_{O_X} \mathcal{E}^{(j)}). \tag{1.7}$$

*Error: having isomorphic stalks does not mean that two sheaves are isomorphic; one must construct the sheaf morphism and *then* verify that this morphism is an isomorphism on the stalks.

Notice that $\mathcal{E}_x^{(i)} \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x^{(j)}$ is a free $\mathcal{O}_{X,x}$-module for every $x \in X$ and hence $\mathcal{E}^{(i)} \otimes_{\mathcal{O}_X} \mathcal{E}^{(j)}$ is a locally free $\mathcal{O}_X$-module so it is a valid choice for showing Brauer equivalence between $\mathcal{A} \sim \mathcal{A}''$ and in general between any two Azumaya algebras over X. Finally, we have

$$\begin{aligned}
\mathcal{A} \otimes \underline{\mathrm{End}}(\mathcal{E} \otimes \mathcal{E}'') &\cong \mathcal{A} \otimes \left(\underline{\mathrm{End}}(\mathcal{E}) \otimes \underline{\mathrm{End}}(E'')\right) \\
&\cong \left(\mathcal{A} \otimes (\underline{\mathrm{End}}(\mathcal{E})\right) \otimes \underline{\mathrm{End}}(\mathcal{E}'') \\
&\cong \left(\mathcal{A}' \otimes \underline{\mathrm{End}}(\mathcal{E}')\right) \otimes \underline{\mathrm{End}}(\mathcal{E}'') \\
&\cong \left(\mathcal{A}' \otimes \underline{\mathrm{End}}(\mathcal{E}'')\right) \otimes \underline{\mathrm{End}}(\mathcal{E}') \\
&\cong \left(\mathcal{A}'' \otimes \underline{\mathrm{End}}(\mathcal{E}''')\right) \otimes \underline{\mathrm{End}}(\mathcal{E}') \\
&\cong \mathcal{A}'' \otimes \mathrm{End}(\mathcal{E}''' \otimes \mathcal{E}'),
\end{aligned}$$

and thus $\mathcal{A} \sim \mathcal{A}''$.

Definition 3. Let X be a scheme. Then the set of equivalence classes of Azumaya algebras over X is denoted by $\mathrm{Br}(X)$. Given two elements $[\mathcal{A}], [\mathcal{A}'] \in \mathrm{Br}(X)$, we define

$$[\mathcal{A}] \cdot [\mathcal{A}'] = [\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}']. \tag{1.8}$$

Remark 1.6. The operation $\cdot$ defined above is well-defined. Indeed, if $\mathcal{A}$ and $\mathcal{A}'$ are Azumaya algebras, then for every $x \in X$, we have

$$\begin{aligned}
(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}')_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) &\cong (\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}'_x) \otimes_{\mathcal{O}_{X,x}} \kappa(x), \qquad \text{by Proposition A.12.(i)}, \\
&\cong \left(\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)\right) \otimes_{\mathcal{O}_{X,x}} \left(\mathcal{A}'_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)\right).
\end{aligned}$$

In other words, $(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}')_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a tensor product of central simple algebras over $\kappa(x)$. It is well known that these are again central simple algebras over $\kappa(x)$ (see for example [Kna07, Proposition 2.36 of Chapter II, §7]) and thus by Theorem 1.3 we have that $\mathcal{A} \otimes \mathcal{A}'$ is again an Azumaya algebra over X. This means that $[\mathcal{A} \otimes \mathcal{A}']$ is an element of $\mathrm{Br}(X)$.

Next, to finish proving that the operation is well-defined, we must show that if $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{A}' \sim \mathcal{B}'$, then $\mathcal{A} \otimes \mathcal{A}' \sim \mathcal{B} \otimes \mathcal{B}'$. Indeed, if

$$\mathcal{A} \otimes \underline{\mathrm{End}}(\mathcal{E}) \cong \mathcal{B} \otimes \underline{\mathrm{End}}(\mathcal{E}') \quad \text{and} \quad \mathcal{A}' \otimes \underline{\mathrm{End}}(\mathcal{E}'') \cong \mathcal{B}' \otimes \underline{\mathrm{End}}(\mathcal{E}''')$$

then we have

$$\begin{aligned}
(\mathcal{A} \otimes \mathcal{A}') \otimes \left(\underline{\mathrm{End}}(\mathcal{E} \otimes \mathcal{E}'')\right) &\cong (\mathcal{A} \otimes \mathcal{A}') \otimes \left(\underline{\mathrm{End}}(\mathcal{E}) \otimes \underline{\mathrm{End}}(\mathcal{E}'')\right) \quad \text{by (1.7)} \\
&\cong \left(\mathcal{A} \otimes \underline{\mathrm{End}}(\mathcal{E})\right) \otimes \left(\mathcal{A}' \otimes \underline{\mathrm{End}}(\mathcal{E}'')\right) \\
&\cong \left(\mathcal{B} \otimes \underline{\mathrm{End}}(\mathcal{E}')\right) \otimes \left(\mathcal{B}' \otimes \underline{\mathrm{End}}(\mathcal{E}''')\right) \quad \text{since } \mathcal{A} \sim \mathcal{B},\ \mathcal{A}' \sim \mathcal{B}' \\
&\cong (\mathcal{B} \otimes \mathcal{B}') \otimes \left(\underline{\mathrm{End}}(\mathcal{E}') \otimes \underline{\mathrm{End}}(\mathcal{E}''')\right) \\
&\cong (\mathcal{B} \otimes \mathcal{B}') \otimes \left(\underline{\mathrm{End}}(\mathcal{E}' \otimes \mathcal{E}''')\right) \quad \text{by (1.7)}, \\
\therefore \qquad \mathcal{A} \otimes \mathcal{A}' &\sim \mathcal{B} \otimes \mathcal{B}'
\end{aligned}$$

We conclude that the operation defined in (1.8) is well-defined.

Proposition 1.7. *The set* $\mathrm{Br}(X)$ *with the operation defined in* (1.8) *is an abelian group with identity element* $[\mathcal{O}_X]$ *and inverses* $[\mathcal{A}]^{-1} = [\mathcal{A}^{\mathrm{op}}]$.

Proof. The operation is associative and commutative because the tensor product is associative and commutative (up to unique isomorphism). The trivial $\mathcal{O}_X$-module $\mathcal{O}_X$ is the identity because $\mathcal{F} \otimes \mathcal{O}_X \cong \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Finally, by Theorem 1.3 we have that

$$[\mathcal{A}] \cdot [\mathcal{A}^{\mathrm{op}}] = [\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}}] = [\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})]. \tag{1.9}$$

However, since

$$\mathcal{O}_X \otimes \underline{\mathrm{End}}(\mathcal{A}) \cong \underline{\mathrm{End}}(\mathcal{A}) \otimes \mathcal{O}_X \cong \underline{\mathrm{End}}(\mathcal{A}) \otimes \underline{\mathrm{End}}(O_X)$$

and $\mathcal{A}$ is locally free, we may take $\mathcal{E} = \mathcal{A}$ and $\mathcal{E}' = \mathcal{O}_X$ to conclude that $\mathcal{O}_X \sim \underline{\mathrm{End}}(\mathcal{A})$ and thus (1.9) reduces to $[\mathcal{A}] \cdot [\mathcal{A}^{\mathrm{op}}] = [\mathcal{O}_X]$ as required. ☐

Proposition 1.8. *The Brauer group defines a contravariant functor from the category of noetherian schemes to the category of abelian groups as follows:*

$$X \mapsto \mathrm{Br}(X) \quad \textit{and} \quad \left(Y \xrightarrow{f} X\right) \mapsto \begin{pmatrix} \mathrm{Br}(X) \xrightarrow{\mathrm{Br}(f)} \mathrm{Br}(Y) \\ [\mathcal{A}] \mapsto [f^*\mathcal{A}] \end{pmatrix}$$

where $f^*\mathcal{A}$ *is the* inverse image *of* $\mathcal{A}$.

Proof. First we show that $\mathrm{Br}(f)$ is well-defined. We show that $f^*\mathcal{A}$ is an Azumaya algebra over Y so $[f^*\mathcal{A}]$ makes sense and then we show that $\mathrm{Br}(f)$ is independent of the choice of representative.

Suppose that $\mathcal{A}$ is an Azumaya algebra over X. Since $\mathcal{A}$ is coherent, then $f^*\mathcal{A}$ is coherent because both X and Y are noetherian (see Proposition 5.8 of Chapter II of [Har77]). Furthermore, by Corollary A.13 we have that for any $y \in Y$:

$$(f^*\mathcal{A})_y \cong \mathcal{A}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y}.$$

Notice that $\mathcal{A}_{f(y)}$ is Azumaya over $\mathcal{O}_{X,f(y)}$ by assumption. So Proposition 1.1.(b) implies that $(f^*\mathcal{A})_y$ is Azumaya over $\mathcal{O}_{Y,y}$ and thus $f^*\mathcal{A}$ is Azumaya over Y. Observe that this argument, together with Proposition A.16.(i), proves:

$$\mathcal{E} \text{ is a locally free } \mathcal{O}_X\text{-module} \quad \Longrightarrow \quad f^*\mathcal{E} \text{ is a locally free } \mathcal{O}_Y\text{-module}.$$

Next, suppose that $\mathcal{A} \sim \mathcal{A}'$ as Azumaya algebras over X. Then there exist locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}').$$

Now, both the tensor product and the $\underline{\mathrm{Hom}}$ functor commute with the pullback functor of a sheaf (cf. Corollary A.15), so

$$f^*\left(\mathcal{A} \otimes_{\mathcal{O}_X} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right) \cong f^*\mathcal{A} \otimes_{\mathcal{O}_Y} f^*\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \cong f^*\mathcal{A} \otimes_{\mathcal{O}_Y} \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f^*\mathcal{E}, f^*\mathcal{E})$$

and analogously for $\mathcal{A}'$ and $\mathcal{E}'$. Putting these isomorphisms together yields

$$f^*\mathcal{A} \otimes_{\mathcal{O}_Y} \underline{\mathrm{End}}_{\mathcal{O}_Y}(f^*\mathcal{E}) \cong f^*\left(\mathcal{A} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})\right) \cong f^*\left(\mathcal{A}' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}')\right) \cong f^*\mathcal{A}' \otimes_{\mathcal{O}_Y} \underline{\mathrm{End}}_{\mathcal{O}_Y}(f^*\mathcal{E}')$$

and hence $f^*\mathcal{A} \sim f^*\mathcal{A}'$ as required.

Finally, since f^* is a contravariant functor from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_Y$-modules, then $\mathrm{Br}(-)$ is functorial. Indeed:

$$\mathrm{Br}(\mathrm{id}_X)([\mathcal{A}]) = [\mathrm{id}_X^*\mathcal{A}] = [\mathrm{id}_X\mathcal{A}] = [\mathcal{A}] \quad \Longrightarrow \quad \mathrm{Br}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Br}(X)}$$
$$\mathrm{Br}(f \circ g)([\mathcal{A}]) = [(f \circ g)^*\mathcal{A}] = [g^*f^*\mathcal{A}] = \mathrm{Br}(g)([f^*\mathcal{A}]) = \mathrm{Br}(g)\mathrm{Br}(f)([\mathcal{A}]) \quad \Longrightarrow \quad \mathrm{Br}(f \circ g) = \mathrm{Br}(g)\mathrm{Br}(f).$$

□

Chapter 2

The Brauer Group of an Affine Scheme

In this chapter, we will describe the Brauer group of X as a subgroup of the Brauer group of its field of fractions. Throughout this chapter, we fix the following notation, though we will regularly add more requirements when needed.

- R is an integral domain.
- K is the fraction field of R and we view R as a subring of K.
- $X = \mathrm{Spec} R$ and $Y = \mathrm{Spec} K$. The inclusion $R \hookrightarrow K$ gives us a morphism $\iota : Y \to X$.

It is well known that the category of coherent $\mathcal{O}_X$-modules is equivalent to the category of finitely generated $R = \mathcal{O}_X(X)$-modules via $M \mapsto \widetilde{M}$ with inverse $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, the "global sections" functor (see for example Corollary 5.5 in Chapter II of [Har77]). Thus, if $\mathcal{A}$ is an Azumaya algebra over X, then $\mathcal{A} \cong \widetilde{A}$ for some Azumaya algebra A over R. Furthermore, under the functor $\mathrm{Br}(-)$ we have

$$\mathrm{Br}(\iota) : \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y) \quad \text{with} \quad [\mathcal{A}] = [\widetilde{A}] \mapsto [\iota^* \widetilde{A}] = [\widetilde{A \otimes_R K}]$$

and thus we obtain the classical group homomorphism

$$\mathrm{Br}(R) \longrightarrow \mathrm{Br}(K) \quad \text{defined by} \quad [A] \mapsto [A \otimes_R K].$$

It is worth remarking what Brauer equivalence looks like the affine case. More specifically, the proof of Proposition 1.7, tells us that $[\mathcal{A}]$ is the trivial class in $\mathrm{Br}(X)$ if and only if $\mathcal{A} \cong \underline{\mathrm{End}}(\mathcal{E})$ for some locally free $\mathcal{O}_X$-module $\mathcal{E}$ on X. Thus, if we are able to find a finitely generated R-module M which is also locally free, i.e. $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p} \subset R$, for which $A \cong \mathrm{End}_R(M)$ as R-algebras, then we would have that $\mathcal{A} \cong \underline{\mathrm{End}}(\widetilde{M})$. This follows from the fact that the functor $M \mapsto \widetilde{M}$ is fully faithful and hence then $\mathrm{End}_R(M) \cong \mathrm{End}_{\mathcal{O}_X}(\widetilde{M})$.

In conclusion, we have that if $\mathcal{A} = \widetilde{A}$, then

$$[A] = 0 \in \mathrm{Br}(R) \quad \Longleftrightarrow \quad A \cong \mathrm{End}_R(M) \text{ for some finitely generated } R\text{-module that is locally free.} \tag{2.1}$$

We now restrict our attention to this setting.

The main goal of this section is to prove the following theorem only for rings of dimension 1.

Theorem 2.1. *Let R be a noetherian integral domain that is integrally closed in its fraction field. Then the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is injective.*

The proof will require a close look at the properties of Azumayas algebras A as R-subalgebras of $A \otimes_R K$. More precisely, Azumaya algebras are *maximal R-orders* in $A \otimes_R K$ and this maximality is what will allow us to compute the triviality of the Brauer class $[A]$ by embedding A into an endomorphism ring.

We will require the study of lattices and orders inside R-algebras that need not be commutative. This nonabelian case slightly complicates the classical computations of orders in commutative algebras such as number fields. After setting up the necessary theory of orders, we will be in a position to prove Theorem 2.1.

2.1 Lattices

In addition to the notation established at the beginning of the chapter, let V be a finite dimensional K-vector space. In particular, V is an R-module via restriction of scalars.

Definition 4. Let Λ be an R-submodule of V. We say that Λ is an *R-lattice* if it satisfies

(i) Λ contains a K-basis of V,

(ii) There exists a finitely generated R-submodule Λ' of V such that $\Lambda \subseteq \Lambda'$.

Remark 2.2. Condition (ii) allows for an R-lattice Λ to not be finitely generated as an R-module. Though most of the lattices we will study will in fact be finitely generated themselves.

Examples 2.3. (of R-lattices)

1. Let Λ be a free R-module of rank equal to $\dim_K(V)$. We can embed Λ into V by mapping a free generating set of Λ onto a basis. It is clear that the image of Λ is an R-lattice.

2. Let M be a finitely generated torsion-free R-module, then M is an R-lattice in $K \otimes_R M$.

   Proof. Since M is torsion-free, by definition, the localization map $M \to S^{-1}M \cong K \otimes_R M$, where $S = R \setminus \{0\}$, is injective. So we identify $m \in M$ with $1 \otimes m \in K \otimes_R M$. Now, if $m_1, \ldots, m_n$ generate M over R, then $1 \otimes m_1, \ldots, 1 \otimes m_n$ generate $K \otimes_R M$ over K. Indeed, if $x \otimes m \in K \otimes_R M$ is any simple tensor, there exist $r_1, \ldots, r_n \in R$ such that $m = r_1 m_1 + \cdots + r_n m_n$ and hence

   $$x \otimes m = x \otimes \left(\sum r_i m_i\right) = \sum x \otimes (r_i m_i) \sum (x r_i)(1 \otimes m_i).$$

   Since M contains $\{1 \otimes m_1, \ldots, 1 \otimes m_n\}$, which in turn contains a K-basis of $K \otimes_R M$, then M satisfies property (i). Since M is itself finitely generated, it immediately satisfies property (ii). □

3. If Λ is an R-lattice in V, then $\mathrm{End}_R(\Lambda)$ is an R-lattice in $\mathrm{End}_K(V)$. See Proposition 2.6 below.

4. (from Number Theory) Let R be an integrally closed noetherian domain and L/K a finite separable extension. Then the integral closure S of R in L is an R-lattice in L.

Since lattices need not be finitely generated R-modules, it is useful to characterize them in terms of nicer R-modules.

Proposition 2.4. *Let Λ be an R-submodule of V. Then, the following are equivalent:*

(i) *Λ is an R-lattice of V.*

(ii) *There exist free R-submodules F_1 and F_2 of V such that*

$$F_1 \subseteq \Lambda \subseteq F_2 \quad \textit{and} \quad \mathrm{rk}(F_1) = \mathrm{rk}(F_2) = \dim_K V$$

Proof.

(i) $\Longrightarrow$ (ii) Let $\{v_1, \ldots, v_n\} \subset \Lambda$ be a K-basis of V. Then

$$F_1 := \bigoplus_{i=1}^{n} R v_i \subseteq \Lambda$$

since Λ is an R-submodule of K. Clearly, F_1 is free of rank $n = \dim_K V$. Next, we construct F_2.

Since Λ is contained in some finitely generated R-submodule N of V there are elements $x_1, \ldots, x_m \in N \subset V$ that generate N over R. Since each $x_j \in V$, we may express it as a K-linear combination of the above basis. More precisely, since K is the fraction field of R, then for every $j = 1, \ldots, m$ we may write

$$x_j = \sum_{i=1}^{n} \frac{r_{ij}}{s_{ij}} v_i \qquad \text{with} \quad r_{ij} \in R, \;\; s_{ij} \in R \setminus \{0\}.$$

So if we multiply all the denominators, say $s = \prod_{ij} s_{ij} \in R \setminus \{0\}$, and define

$$F_2 := s^{-1} F_1 = \bigoplus_{i=1}^{v} R(s^{-1} v_i)$$

then by construction $x_i \in F_2$ for all i so that $N \subseteq F_2$ and hence $\Lambda \subseteq F_2$ as required.

$(ii) \implies (i)$ Suppose that Λ is between two free R-submodules of V, both of rank $n = \dim_K(V)$. The second property of being an R-lattice is automatic by taking $\Lambda' = F_2$. Now, since F_1 is free, then

$$KF_1 = K \otimes_R F_1 \cong K \otimes_R R^n \cong (K \otimes_R R)^n \cong K^n.$$

So KF_1 is an n-dimensional vector space so it contains a K-basis, say $\{x_1, \dots, x_n\}$. Then, each x_i may be written as $x_i = k_i f_i$ with $k_i \in K$ and $f_i \in F_1$. Notice that $k_i \neq 0$ since otherwise $\{x_1, \dots, x_n\}$ wouldn't be linearly independent. In particular, the set $\{k_1^{-1}x_1, \dots, k_n^{-1}x_n\} \subset F_1 \subset \Lambda$ is linearly independent in V and hence a K-basis. This shows that Λ is an R-lattice.

□

Next, we study an important class of lattices called *reflexive* lattices. These are modules that are isomorphic to their double dual module. This property is key in proving that $\mathrm{Br}(R)$ embeds into $\mathrm{Br}(K)$. First,

Definition 5. Let V and V' be vector spaces over K. Let Λ and Λ' be R-lattices in V and V' respectively. The *lattice quotient* of Λ and Λ' is the R-module:

$$(\Lambda' : \Lambda) = \{f \in \mathrm{Hom}_K(V, V') \mid f(\Lambda) \subseteq \Lambda'\}.$$

Remarks 2.5. (about lattice quotients)

1. The lattice quotient satisfies the following inclusion property:

$$\Lambda_1 \subseteq \Lambda_2 \quad \text{and} \quad \Lambda_1' \subseteq \Lambda_2' \quad \implies \quad (\Lambda_1' : \Lambda_2) \subseteq (\Lambda_2' : \Lambda_1).$$

In fact, this explains the "ratio" notation since the classical ratio $x : y$ of two numbers becomes larger if x (resp. y) is replaced by a larger (resp. smaller) number.

Proof. If $f \in (\Lambda_1' : \Lambda_2)$, then $f(\Lambda_1) \subseteq f(\Lambda_2) \subseteq \Lambda_1' \subseteq \Lambda_2'$. □

2. The restriction map is an R-module isomorphism:

$$(\Lambda' : \Lambda) \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, \Lambda') \quad \text{defined by} \quad f \mapsto f|_\Lambda.$$

Hence we will identify $(\Lambda' : \Lambda)$ with $\mathrm{Hom}_R(\Lambda, \Lambda')$.

Proof. Notice that if $f \in (\Lambda' : \Lambda)$, then by definition, the image of $f|_\Lambda$ lies in Λ'. Furthermore, since f is K-linear, it is R linear and hence $f|_L \in \mathrm{Hom}_R(\Lambda, \Lambda')$. It is clearly an R-module homomorphism. Next, if $f|_\Lambda = 0$, then $f|_\Lambda$ is zero on the K-basis contained in Λ and hence $f = 0$ as an element of $\mathrm{Hom}_K(V, V')$; this shows that the restriction map is injective. Finally, if $f \in \mathrm{Hom}_R(\Lambda, \Lambda')$, then tensoring with K yields

$$\mathrm{id}_K \otimes f : K \otimes_R \Lambda \longrightarrow K \otimes_R \Lambda'.$$

Since Λ is an R-submodule of V, a K-vector space, it is torsion free and hence embeds into $K \otimes_R \Lambda$. By the functoriality of tensoring with K, we have the commutative diagram:

$$\begin{array}{ccc} \Lambda & \hookrightarrow & K \otimes_R \Lambda \\ \downarrow f & & \downarrow \mathrm{id}_K \otimes f \\ \Lambda' & \hookrightarrow & K \otimes_R \Lambda' \end{array}$$

Since the canonical homomorphism $\Lambda \hookrightarrow K \otimes_R \Lambda$ is just multiplication by K, we have $K \otimes_R \Lambda = K\Lambda = V$ so we may view $\mathrm{id}_K \otimes f \in \mathrm{Hom}_K(V, V')$. On the other hand $(\mathrm{id}_K \otimes f)|_\Lambda$ is just the composition of the top arrow with the right arrow, so by commutativity of the diagram $(\mathrm{id}_K \otimes f)|_\Lambda = f$ and we conclude that the restriction map is is surjective. □

3. Let S be a multiplicative subset of R. Since K is the fraction field of R, we view the localization $S^{-1}R$ as a subring of K. Then

$$S^{-1}(\Lambda' : \Lambda) = (S^{-1}\Lambda' : S^{-1}\Lambda)$$

Proof. Let $f/s \in S^{-1}(\Lambda' : \Lambda)$. Then for every $\lambda/t \in S^{-1}\Lambda$ we have

$$(f/s)(\lambda/t) = \frac{1}{s} f(\lambda/t) = \frac{1}{st} f(\lambda) \in S^{-1}\Lambda'$$

since $f(\lambda) \in \Lambda'$. This shows that $f/s \in (S^{-1}\Lambda' : S^{-1}\Lambda)$ Conversely, let $f \in (S^{-1}\Lambda' : S^{-1}\Lambda)$, then for a generating set $\{\lambda_1, \dots, \lambda_n\}$ of some finitely generated R-submodule M of V containing Λ, we have that $f(\lambda_i/1) \in S^{-1}(\Lambda')$ for every $i = 1, \dots, n$, so there exists $\lambda_i' \in \Lambda'$ and $s_i \in S$ such that $f(\lambda_1/1) = \lambda_i'/s_i'$. If we define $s = s_1 \cdots s_n$, then $(sf)(\lambda_i) \in R\Lambda' = \Lambda'$ and hence $sf \in (\Lambda' : \Lambda)$ so that $f = sf/s \in S^{-1}(\Lambda' : \Lambda)$. □

The reason $(\Lambda' : \Lambda)$ is called the quotient *lattice* is because it is one.

Proposition 2.6. *Let V and V' be vector spaces over K. Let Λ and Λ' be R-lattices in V and V' respectively. Then*

$$(\Lambda' : \Lambda) = \mathrm{Hom}_R(\Lambda, \Lambda') \text{ is an } R\text{-lattice in } \mathrm{Hom}_K(V, V').$$

Proof. First we prove the case when Λ and Λ' are *free* R-lattices and then use Proposition 2.4 to finish the proof. Under this strong assumption we have that $(\Lambda' : \Lambda) = \mathrm{Hom}_R(\Lambda, \Lambda')$ is a free R-module and hence flat, which means that tensoring the inclusion $R \hookrightarrow K$ with $\mathrm{Hom}_R(\Lambda, \Lambda')$ preserves injectivity. Thus, after an application of Proposition A.7, we obtain the embedding

$$\mathrm{Hom}_R(\Lambda, \Lambda') \cong R \otimes_R \mathrm{Hom}_R(\Lambda, \Lambda') \hookrightarrow K \otimes_R \mathrm{Hom}_R(\Lambda, \Lambda') \cong \mathrm{Hom}_K(K \otimes_R \Lambda, K \otimes_R \Lambda') \cong \mathrm{Hom}_K(V, V).$$

Since $\mathrm{Hom}_R(\Lambda, \Lambda')$ is free of rank

$$\mathrm{rk}(\Lambda) \times \mathrm{rk}(\Lambda') = \dim_K(V) \dim_K(V') = \dim_K(\mathrm{Hom}_K(V, V')).$$

By taking $F_1 = F_2 = \mathrm{Hom}_R(\Lambda, \Lambda')$ in Proposition 2.4, $\mathrm{Hom}_R(\Lambda, \Lambda')$ is an R-lattice.

Next, we relax the freeness assumption on Λ and Λ'. By Proposition 2.4, there are free R-submodules $F_1, F_2 \subseteq V$ and $F_1', F_2' \subseteq V'$ such that

$$\begin{aligned} F_1 \subseteq \Lambda \subseteq F_2 \quad &\text{and} \quad \mathrm{rk}(F_1) = \mathrm{rk}(F_2) = \dim_K V, \\ F_1' \subseteq \Lambda' \subseteq F_2' \quad &\text{and} \quad \mathrm{rk}(F_1') = \mathrm{rk}(F_2') = \dim_K V'. \end{aligned}$$

Thus by the inclusion properties of the quotient lattice (see Remark 2.5.1) we have

$$(F_1' : F_2) \subseteq (\Lambda' : \Lambda) \subseteq (F_2' : F_1).$$

By the previous part, we saw that $(F_1' : F_2)$ and $(F_2' : F_1)$ are both free R-lattices of rank $\dim_K(\mathrm{Hom}_K(V, V'))$, then we can use Proposition 2.4 again to conclude that $(\Lambda' : \Lambda)$ is an R-lattice.

□

Example 2.7. (Dual lattice) If we set $V' = K$ and $\Lambda' = R$, then

$$(R : \Lambda) = \mathrm{Hom}_R(\Lambda, R) = \Lambda^*$$

So the dual module of Λ is a quotient lattice. In particular,

Λ^* is an R-lattice in the vector space dual V^*.

Example 2.8. (Double-Dual lattice) If we set $V' = V^*$ and $\Lambda' = \Lambda^*$, which we can do by the previous example, then

$$(R : \Lambda^*) = (R : \mathrm{Hom}_R(\Lambda, R)) = \mathrm{Hom}_R(\mathrm{Hom}_R(\Lambda, R), R) = \Lambda^{**}$$

So the double dual module of Λ is a quotient lattice and

Λ^{**} is an R-lattice in the vector space double dual V^{**}.

Next we study reflexive lattices. These are those that are isomorphic to their double dual module. Analogously as in the case of vector spaces, we have the R-module homomorphism:

$$\Lambda \hookrightarrow \Lambda^{**} \quad \text{defined by} \quad \lambda \mapsto (f \mapsto f(\lambda))$$

By the functoriality of the hom functor, we obtain the commutative diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda^{**} \\ \big\downarrow & & \big\downarrow \\ V & \xrightarrow{\sim} & V^{**} \end{array}$$

where the vertical arrows are the inclusions of lattices into their ambient vector spaces and the bottom arrow is the canonical isomorphism. By commutativity, the top arrow is injective.

Definition 6. Let Λ be an R-lattice in V. Then Λ is *reflexive* if the canonical embedding $\Lambda \hookrightarrow \Lambda^{**}$ is an isomorphism.

The main reason we are interested in reflexive lattices is that in some cases, reflexive lattices are projective and hence give us candidates for showing that Azumaya algebras have trivial Brauer class. More precisely we have a theorem due to Auslander and Goldman:

Theorem 2.9. *Let R a neotherian regular domain with fraction field K. Let Λ an R-lattice in some K-vector space V. Then*

$$\Lambda \text{ is reflexive and } \operatorname{End}_R(\Lambda) \text{ is a projective } R\text{-module} \quad \Longrightarrow \quad \Lambda \text{ is a projective } R\text{-module.}$$

While the proof of this theorem lies outside of the scope of these notes, we can prove a weaker version that replaces the condition on $\operatorname{End}_R(\Lambda)$ with a restriction on the dimension of R.

Theorem 2.10. *Let R be a regular noetherian domain of Krull dimension $\dim R \leq 1$. Let Λ be an R-lattice in V that is also finitely generated as an R-module. Then*

$$\Lambda \text{ is reflexive} \quad \Longrightarrow \quad \Lambda \text{ is projective.}$$

Proof. We separate the proof into three cases, one for each possible dimension.

($\dim R = 0$) Dimension zero rings are fields. Thus, Λ is automatically free and hence projective.

($\dim R = 1$) By the characterization of finitely generated projective modules, cf. Proposition A.9, we must show that Λ is finitely presented and that $\Lambda_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ of R. The former property follows immediately from our assumptions that R is noetherian and Λ is finitely generated (see Remark A.10).

To show that $\Lambda_\mathfrak{m}$ is free, we will apply the structure theorem of finitely generated modules over a PID. To do this we must verify that $\Lambda_\mathfrak{m}$ is finitely generated, torsion-free and that $R_\mathfrak{m}$ is a PID. The first two properties are easy to verify. Since Λ is finitely generated as an R-module, then $\Lambda_\mathfrak{m}$ is finitely generated as an $R_\mathfrak{m}$-module; just tensor the exact sequence $R^n \twoheadrightarrow \Lambda \to 0$ with the flat R-module $R_\mathfrak{m}$ to get an epimorphism $R_\mathfrak{m}^n \twoheadrightarrow \Lambda_\mathfrak{m}$. Next, since Λ is torsion-free as an R-module (it is contained in a K-vector space), then $\Lambda_\mathfrak{m}$ is torsion-free as an $R_\mathfrak{m}$-module as it is also contained in the same K-vector space as Λ. Thus, if $R_\mathfrak{m}$ is a PID, then we may apply the structure theorem of finitely generated torsion-free modules over a PID and conclude that $\Lambda_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module as required.

The last step in this proof is to show that $R_\mathfrak{m}$ is a PID. Since R is a regular noetherian domain, then $R_\mathfrak{m}$ is a regular noetherian local ring by definition. Now, $\dim R_\mathfrak{m} = \operatorname{ht}\mathfrak{m} \leq \dim R = 1$ by definition. Since $\mathfrak{m}$ is maximal and R is a not a field, then $\operatorname{ht}\mathfrak{m} = 1$. Thus $R_\mathfrak{m}$ is a regular local domain of dimension 1. This is equivalent to being a PID by Proposition A.3. This completes the proof.

□

2.2 Orders

Definition 7. Let A be a K-algebra that is finite dimensional as a K-vector space. Then $\mathcal{O} \subseteq A$ is called an *R-order* if it satisfies

(*i*) $\mathcal{O}$ is an R-subalgebra of A,

(*ii*) $\mathcal{O}$ is an R-lattice of A.

Theorem 2.11. *Let R be a noetherian integrally closed domain and K its field of fractions. Let A be a central simple algebra over K. Let $\mathcal{O} \subset A$ be an R-subalgebra of A such that $K\mathcal{O} = A$. Then*

$$\mathcal{O} \text{ is an } R\text{-order of } A \quad \Longleftrightarrow \quad \mathcal{O}/R \text{ is an integral extension.}$$

Proof. We prove a double implication:

($\Longrightarrow$) Let $\alpha \in \mathcal{O}$. We want to show that α is integral over R. By definition of lattice, there is a finitely generated R-submodule $\mathcal{O}'$ of A such that $\mathcal{O} \subseteq \mathcal{O}'$; in particular, $R[\alpha] \subseteq \mathcal{O}'$. This means that the multiplication map $x \mapsto \alpha x$ is an R-linear endomorphism ψ of $\mathcal{O}'$ and hence by Cayley-Hamilton must satisfy a *monic* equation:

$$\psi^n + r_{n-1}\psi^{n-1} + \cdots + r_1\psi + r_0 \qquad (r_i \in R).$$

Applying both sides of this equation to $1 \in \mathcal{O}$ yields an integral dependence equation for α.

($\Longleftarrow$) Since $\mathcal{O}$ is already an R-subalgebra of A by assumption (and hence an R-submodule), we must show that $\mathcal{O}$ is an R-lattice of A. We prove each property separately:

(i) Let $\{a_1, \ldots, a_n\} \subset A$ be a K-basis of A. Since $K\mathcal{O} = A$ by assumption, then there exist $\mu_1, \ldots, \mu_n \in K$ and $\lambda_1, \ldots, \lambda_n \in \mathcal{O}$ such that $a_i = \mu_i \lambda_i$. It is clear that each $\mu_i \neq 0$ so that $\{\lambda_1, \ldots, \lambda_n\}$ is also a K-basis of A and it is contained in $\mathcal{O}$.

(ii) See Lemma 11.3.2 of [For17].

□

Finally, we finish the section by putting everything together and applying the theory of R-orders to Azumaya algebras over R.

Theorem 2.12. *Let R be an integrally closed domain and let A an Azumaya algebra over R. Then A is a maximal R-order in $A \otimes_R K$.*

Proof. For simplicity, we write $B := A \otimes_R K$, which is an R-algebra. Since A is an Azumaya algebra, it is a finitely generated flat R-module. Thus tensoring with A over R embeds A into B as an R-lattice by Example 2.3.2. Since A is itself an R-algebra, then A is an R-order in B; as in Example 2.3.2 we also make the identification $B = AK$ and view A as a subalgebra of B. We recall the fact that B is a central simple algebra over K by virtue of A being an Azumaya algebra over R (see Proposition 1.1.(a)) and hence $Z(B) = K$.

Next, let $\mathcal{O} \subseteq B$ be an R-order that contains A and consider the centralizer of A in $\mathcal{O}$, i.e.

$$Z := Z_{\mathcal{O}}(A) = \{z \in \mathcal{O} \mid za = az, \ \ \forall a \in A\} \subseteq \mathcal{O}.$$

Notice that Z is an R-subalgebra of $\mathcal{O}$. If $b = ak \in B = AK$ is an arbitrary element, with $a \in A$ and $k \in K$, then if $z \in Z$ we have

$$zb = z(ak) = (za)k \overset{*}{=} k(za) = k(az) = (ka)z \overset{*}{=} (ak)z = bz,$$

where the equalities marked by $*$ follow from the fact that $Z(B) = K$. We have shown that every element of Z commutes with every element of B, i.e. $Z \subseteq K$ and thus

$$Z \subseteq K \cap \mathcal{O}.$$

However, Theorem 2.11 tells us that every element of $\mathcal{O}$, and hence of Z, is integral over R. By assumption, R is integrally closed in K, so that $Z \subseteq R$ and thus $Z = R$.

Finally, in the proof of the double centralizer theorem for central simple algebras, it is shown that the multiplication map

$$A \otimes_R Z \longrightarrow \mathcal{O} \quad \text{defined by} \quad a \otimes z \mapsto az$$

is a natural R-algebra isomorphism (see for example §8 of [Kna07]). We may thus identify $\mathcal{O}$ with AZ so that $\mathcal{O} = AZ = AR = A$. This proves that A is a maximal R-order in B as required. □

2.3 The Brauer Group of a Regular Ring

The main goal of this section is to prove the following theorem:

Theorem 2.13. *Let R be a neotherian regular domain of dimension 1 with fraction field K. Then the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is injective.*

Proof. Let A be an Azumaya algebra over R such that its image $[A \otimes_R K]$ is trivial in $\mathrm{Br}(K)$, that is $B := A \otimes_R K \cong M_{n \times n}(K)$ as K-algebras for some $n > 0$. Furthermore, Theorem 2.12 states that A is a maximal R-*order* in B.

Now, let V be an n-dimensional K-vector space (e.g. $V = K^n$) so we may identify B with the K-algebra $\mathrm{End}_K(V)$ which makes V into a B-module via $\varphi \cdot v := \varphi(v)$. Next we fix a nonzero vector $v \in V \setminus \{0\}$. The evaluation map

$$\epsilon : B \longrightarrow V \quad \text{defined by} \quad \varphi \mapsto \varphi(v),$$

is clearly surjective since given any nonzero vector w of V, a change of basis matrix sending v to w is an endomorphism that maps onto w. Furthermore, it is a B-module homomorphism since for any $\varphi_1, \varphi_2 \in B$ we have

$$\epsilon(\varphi_1 \circ \varphi_2) = (\varphi_1 \circ \varphi_2)(v) = \varphi_1(\varphi_2(v)) = \varphi_1(\epsilon(\varphi_2)) = \varphi_1 \cdot \epsilon(\varphi_2).$$

Next we restrict ϵ to $A \subset B$ and corestrict to its image. More precisely, define $\Lambda := \epsilon(A)$ to be the image of A under ϵ and

$$\varepsilon : A \longrightarrow \Lambda \quad \text{defined as} \quad \varepsilon := \epsilon|_A$$

The reason why we use Λ to denote $\epsilon(A)$ is that Λ is an R-lattice in V. To prove that this is indeed the case, we must show three things:

- (Λ is an R-submodule of V) If $w_1, w_2 \in \Lambda$, then there is an $\varphi_1, \varphi_2 \in A$ such that $\varphi_i(v) = w_i$ and since A is an R-subalgebra of the R-module endomorphisms of V, for any $r \in R$ we have

$$rw_1 + w_2 = r\varphi_1(v) + \varphi_2(v) = (r\varphi_1 + \varphi_2)(v) \quad \text{where} \quad r\varphi_1 + \varphi_2 \in A$$

  and thus $rw_1 + w_2 \in \Lambda$; Λ is an R submodule of V.

- (Λ contains a K-basis of V) Since ϵ is an epimorphism, $\dim_K \ker \epsilon = \dim_K B/\dim_K V = n$ and $B/\ker \epsilon$ is an n-dimensional K vector space; let $\{\psi_1 + \ker \epsilon, \ldots, \psi_n + \ker \epsilon\}$ be a basis. Since $B = KA$, we have $\psi_i = \mu_i \varphi_i$, where $\mu_i \in K^\times$ and $\varphi_i \in A$. Notice that

$$\varepsilon(\varphi_i) = \epsilon(\varphi_i) = \epsilon(\mu_i^{-1}\psi_i) = \mu_i^{-1}\psi_i(v),$$

  So that any K-linear combination $k_1\varepsilon(\varphi_1) + \cdots + k_n\varepsilon(\varphi_n) = 0$, induces a K-linear combination

$$\begin{aligned} k_1\varphi_1(v) + \cdots + k_n\varphi_n(v) = 0 \quad &\Longrightarrow \quad k_1\varphi_1 + \cdots + k_n\varphi_n \in \ker \epsilon \\ &\Longrightarrow \quad k_1\mu_1^{-1}(\psi_1 + \ker \epsilon) + \cdots + k_n\mu_n^{-1}(\psi_n + \ker \epsilon) = 0 \end{aligned}$$

  and hence $k_1 = \cdots = k_n = 0$. This shows that $\{\varepsilon(\varphi_1), \ldots, \varepsilon(\varphi_n)\} \subset \Lambda$ is K-linearly independent in V and thus a K-basis.

- (Λ is contained in a finitely generated R-submodule Λ' of V) Since ε is a surjective R-module homomorphism and A is a finitely generated R-module, then Λ itself is finitely generated as an R-module.

Next, we embed A into $\mathrm{End}_R(\Lambda)$ via the left regular representation. More precisely, consider the map

$$\rho : A \longrightarrow \mathrm{End}_R(\Lambda) \quad \text{defined by} \quad \rho(\varphi)(\lambda) = \varphi(\lambda), \quad \lambda \in \Lambda.$$

This is well defined because if $\varphi \in A$ and $\lambda \in \Lambda$, then there exists $\varphi' \in A$ such that $\varepsilon(\varphi') = \varphi'(v) = \lambda$ and hence

$$\rho(\varphi)(\lambda) = \varphi(\lambda) = \varphi(\varphi'(v)) = (\varphi \circ \varphi')(v) = \varepsilon(\varphi \circ \varphi') \in \Lambda.$$

It is routine to check that ρ is an R-algebra homomorphism; below we just verify that it is multiplicative: if $\varphi, \varphi' \in A$ and $r \in R$, then for every $\lambda \in \Lambda$,

$$\rho(\varphi \circ \varphi')(\lambda) = (\varphi \circ \varphi')(\lambda) = \varphi(\varphi'(\lambda)) = \varphi(\rho(\varphi)(\lambda)) = \rho(\varphi)(\rho(\varphi')(\lambda)) = \big(\rho(\varphi) \circ \rho(\varphi')\big)(\lambda).$$

Finally, ρ is injective: if $\varphi \in \ker \rho$, then $\rho(\varphi)(\lambda) = \varphi(\lambda) = 0$ for all $\lambda \in \Lambda$ and in particular φ vanishes on the basis of V that is contained in Λ and hence $\varphi = 0$. This means that we can identify A as a subalgebra of $\mathrm{End}_R(\Lambda)$.

By Proposition 2.6 $\mathrm{End}_R(\Lambda)$ is an R-order of B, so the maximality of A as an R-order, given by Theorem 2.12 implies that $A = \mathrm{End}_R(\Lambda)$. In view of our characterization of $[A] = 0$ from (2.1), we would be close to finishing the proof. However, we cannot guarantee that Λ is locally free. However, this problem can be solved by changing Λ to Λ^{**}.

Lets write $M := \Lambda^{**}$. The map

$$\alpha : \Lambda^* \otimes_R \Lambda \longrightarrow \mathrm{Hom}_R(\Lambda, \Lambda)^* \quad \text{defined by} \quad \alpha(f \otimes \lambda)(g) = f(g(\lambda))$$

is an R-module homomorphism. It is a natural isomorphism whenever Λ is a finitely generated projective R-module. Since we don't have this assumption, we must look at the problem locally.

Every prime ideal $\mathfrak{p} \subset R$ is of height 1 since $\dim R = 1$, so the localization $R_\mathfrak{p}$ is regular (since R is regular by assumption) with $\dim(R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) = 1$. In dimension 1, being regular is equivalent to being a PID, so $R_\mathfrak{p}$ is a PID. This means that $\Lambda_\mathfrak{p} := R_\mathfrak{p} \otimes_R \Lambda$ is a finitely generated torsion-free (since Λ is contained in the K-vector space V) $R_\mathfrak{p}$-module. By the structure theorem for finitely generated modules over a PID, we may conclude that $\Lambda_\mathfrak{p}$ is free. With this in mind, we obtain the following canonical isomorphisms:

$$\begin{aligned} R_\mathfrak{p} \otimes_R (\Lambda^* \otimes_R \Lambda) &\cong (R_\mathfrak{p} \otimes_R \Lambda^*) \otimes_{R_\mathfrak{p}} (R_\mathfrak{p} \otimes_R \Lambda) && \text{base change,} \\ &= (R_\mathfrak{p} \otimes_R \mathrm{Hom}_R(\Lambda, R)) \otimes_{R_\mathfrak{p}} \Lambda_\mathfrak{p} \\ &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\Lambda_\mathfrak{p}, R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \Lambda_\mathfrak{p} && \text{by Proposition A.7,} \\ &\cong \mathrm{Hom}_{R_\mathfrak{p}}\big(\mathrm{Hom}_{R_\mathfrak{p}}(\Lambda_\mathfrak{p}, \Lambda_\mathfrak{p}), R_\mathfrak{p}\big) && \text{by Proposition A.6,} \\ &\cong \mathrm{Hom}_{R_\mathfrak{p}}\big(R_\mathfrak{p} \otimes_R \mathrm{Hom}_R(\Lambda, \Lambda), R_\mathfrak{p} \otimes_R R\big) && \text{by Proposition A.7,} \\ &\cong R_\mathfrak{p} \otimes_R \mathrm{Hom}_R\big(\mathrm{Hom}_R(\Lambda, \Lambda), R\big) && \text{by Proposition A.7,} \\ &= R_\mathfrak{p} \otimes_R \mathrm{Hom}_R(\Lambda, \Lambda)^*. \end{aligned}$$

Thus, the localized maps $\alpha_{\mathfrak{p}} : R_{\mathfrak{p}} \otimes_R (\Lambda^* \otimes \Lambda) \to R_{\mathfrak{p}} \otimes_R \mathrm{Hom}(\Lambda, \Lambda)^*$ are all isomorphisms and thus the dualized maps

$$\alpha_{\mathfrak{p}}^* : \mathrm{Hom}_R(\Lambda, \Lambda)^{**} \longrightarrow (\Lambda^* \otimes_R \Lambda)^*$$

are all isomorphisms. Thus

$$\begin{aligned} \mathrm{Hom}_R(\Lambda, \Lambda)^{**} &\cong (\Lambda^* \otimes_R \Lambda)^* \\ &= \mathrm{Hom}_R(\Lambda^* \otimes_R \Lambda, R) \\ &\cong \mathrm{Hom}_R(\Lambda \otimes_R \Lambda^*, R) \\ &\cong \mathrm{Hom}_R\big(\Lambda^*, \mathrm{Hom}_R(\Lambda, R)\big) && \text{tensor-hom adjunction} \\ &= \mathrm{Hom}_R(\Lambda^*, \Lambda^*) \end{aligned}$$

Since $\mathrm{Hom}_R(\Lambda, \Lambda)^{**}$ is reflexive, we have

$$\mathrm{Hom}_R(\Lambda, \Lambda)^{**} \cong \big(\mathrm{Hom}_R(\Lambda, \Lambda)^{**}\big)^{**} \cong \mathrm{Hom}_R(\Lambda^*, \Lambda^*)^{**} \cong \mathrm{Hom}_R(\Lambda^{**}, \Lambda^{**})$$

Furthermore

$$\mathrm{End}_R(\Lambda) \hookrightarrow \mathrm{End}_R(\Lambda)^{**} = \mathrm{End}_R(\Lambda^{**})$$

so we may view A as a subalgebra of $\mathrm{End}_R(\Lambda^{**})$. By maximality we have $A = \mathrm{End}_R(\Lambda^{**})$. Since Λ^{**} is reflexive, then Theorem 2.10 tells us that Λ^{**} is projective and Proposition A.9 implies that Λ^{**} is locally free. Thus we can now safely conclude that $[A] = 0$ inside the Brauer group. This finishes the proof that the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is injective.

□

Appendix A

Miscellaneous Results

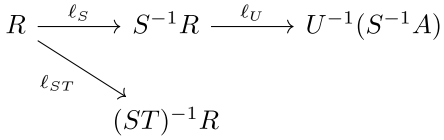
A.1 Tools from Commutative Algebra

A.1.1 Commutative Rings

Proposition A.1. *(Exercise 3 of Chapter III of [AM94]) Let R be a commutative ring and let $S, T \subset R$ be two multiplicatively closed sets. Let $\ell_S : R \to S^{-1}R$ denote the canonical localization map and set $U := \ell(T) \subset S^{-1}R$. Then there is a canonical isomorphism*

$$U^{-1}(S^{-1}R) \cong (ST)^{-1}R.$$

Proof. Note that since R is commutative, ST is multiplicatively closed and hence $(ST)^{-1}R$ makes sense. Also, since ℓ is a ring homomorphism, U is multiplicatively closed and thus $U^{-1}(S^{-1}R)$ also makes sense. By our assumptions, we have the following diagram

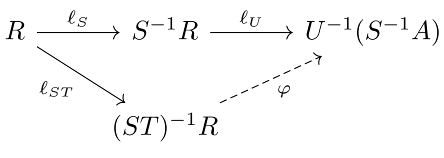


The required isomorphism (and its inverse) will be set up with the universal property of the localization map.

Let $st \in ST$ be an arbitrary element. Then

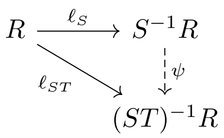
$$\ell_U \ell_S(st) = \ell_U(\ell_S(s)\ell_T(s)) = \ell_U(\ell_S(s)) \cdot \ell_U(\ell_S(t)).$$

Since $s \in S$, we have $\ell_S(s) \in (S^{-1}R)^\times$. Since unital ring homomorphisms preserve units we have that $\ell_U(\ell_S(s)) \in (U^{-1}(S^{-1}R))^\times$. On the other hand, since $t \in T$, we have $\ell_S(t) \in \ell(T) = U$ and thus $\ell_U(\ell_S(t))$ is a unit in $U^{-1}(S^{-1}R)$. These two arguments show that $\ell_U \ell_S(st)$ is a product of units and hence itself a unit. By the universal property of localization, the map $\ell_U \ell_S$ factors through ℓ_{ST}, i.e. there exists a (unique) ring homomorphism $\varphi : (ST)^{-1} \to U^{-1}(S^{-1}A)$ that makes the following diagram commute:



$$\ell_U \ell_S = \varphi \ell_{ST}$$

Next, if $s \in S$ then $s = s \cdot 1 \in ST$ (since $1 \in T$) so that $\ell_{ST}(s)$ is a unit in $(ST)^{-1}R$. By the universal property of localization, there is a (unique) ring homomorphism $\psi : S^{-1}R \to (ST)^{-1}R$ that makes the following diagram commute:



$$\ell_{ST} = \psi \ell_S$$

Combining this with the previous commutative diagram, we have that

$$(\ell_U)\ell_S = \varphi \ell_{ST} = (\varphi\psi)\ell_S.$$

In other words, both ℓ_U and $\varphi\psi$ satisfy the same universal property, namely:

$$\begin{array}{ccc} S^{-1}R & & \\ {\scriptstyle \ell_S}\uparrow & \overset{\ell_U=\varphi\psi}{\searrow} & \\ R & \xrightarrow{\varphi\ell_{ST}} & U^{-1}(S^{-1}R) \end{array}$$

Note that $\varphi\ell_{ST}$ does indeed map S into the units of $U^{-1}(S^{-1}R)$ as shown above. By the uniqueness of the universal property, we have

$$\ell_U = \varphi\psi. \tag{A.1}$$

On the other hand, if $u = \ell_S(s) \in U$ is arbitrary, then by the commutativity of the above diagram we have

$$\psi(\ell_S(s)) = \ell_{ST}(s) = \ell_{ST}(s \cdot 1) \in \big((ST)^{-1}R\big)^{\times}.$$

Thus ψ maps U into the units of $(ST)^{-1}R$ so the universal property of localizations there is a (unique) ring homomorphism $\xi : U^{-1}(S^{-1}A) \to (ST)^{-1}R$ that makes the following diagram commute:

$$\begin{array}{ccc} S^{-1}R & \xrightarrow{\ell_U} & U^{-1}(S^{-1}A) \\ {\scriptstyle \psi}\downarrow & \overset{\xi}{\swarrow} & \\ (ST)^{-1}R & & \end{array} \qquad \psi = \xi\ell_U$$

In fact, if we postcompose $\psi = \xi\ell_U$ with φ and use (A.1) we have:

$$(\varphi\xi)\ell_U = \varphi\psi = \ell_U = \ell_U.$$

In other words, both $\varphi\xi$ and id satisfy the same universal property, namely

$$\begin{array}{ccc} U^{-1}(S^{-1}R) & & \\ {\scriptstyle \ell_U}\uparrow & \overset{\varphi\xi=\mathrm{id}}{\searrow} & \\ S^{-1}R & \xrightarrow{\ell_U} & U^{-1}(S^{-1}R) \end{array}$$

and thus uniqueness implies that $\varphi\xi = \mathrm{id}$.

Conversely, if we postcompose $\ell_U\ell_S = \varphi\ell_{ST}$ with ξ we have

$$(\xi\varphi)\ell_{ST} = \xi\ell_U\ell_S = \psi\ell_S = \ell_{ST}. \tag{A.2}$$

That is, $\xi\varphi$ and the identity satisfy the following universal property

$$\begin{array}{ccc} (ST)^{-1}R & & \\ {\scriptstyle \ell_{ST}}\uparrow & \overset{\xi\varphi=\mathrm{id}}{\searrow} & \\ R & \xrightarrow{\ell_{ST}} & (ST)^{-1}R \end{array} \qquad \exists!\eta \text{ such that } \ell_{ST} = \eta\ell_{ST}$$

and thus $\mathrm{id} = \xi\varphi$. Since both ξ and φ were uniquely given the isomorphism $U^{-1}(S^{-1}R) \cong (ST)^{-1}R$ is unique. This finishes the proof. □

Remark A.2. We explicitly write down what Proposition A.1 looks like for the type of localizations we are interested:

1. If $S = \{f^n \mid n \in \mathbb{Z}\}$ and $T = \{g^n \mid n \in \mathbb{Z}\}$, then

$$R_{fg} \cong (R_g)_{f/1} \cong (R_f)_{g/1}.$$

2. Let $\mathfrak{p} \subset \mathfrak{q}$ be two prime ideals with $S = R \setminus \mathfrak{q}$ and $T = R \setminus \mathfrak{p}$. Since $S \subseteq T$, we have $ST = T$. Furthermore, directly from the definitions we have $U = \ell_S(R \setminus \mathfrak{p}) = R_{\mathfrak{q}} \setminus \mathfrak{p}R_{\mathfrak{q}}$ so we can conclude:

$$R_{\mathfrak{p}} \cong (R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$$

Proposition A.3. *Let R be a noetherian local domain of dimension 1 with (unique) maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Then the following are equivalent:*

(i) *R is regular, i.e.* $\dim R = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$,

(ii) *R is a PID,*

(iii) *R is a DVR.*

Proof. We only prove that $(i) \iff (ii)$ since this is the part we will use in these notes. We refer to Proposition 9.2 of [AM94] for the more complete version of this Proposition and its proof.

$(i \implies ii)$ Since R is regular of dimension 1, then $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$. Let $x \in \mathfrak{m}$ be such that $x+\mathfrak{m}^2$ generates $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space. Since R is noetherian, $\mathfrak{m}$ is a finitely generated R-module so we may apply Nakayama's Lemma to lift the generator; that means $\mathfrak{m} = (x)$ and $\mathfrak{m}$ is principal.

Let $\mathfrak{a} \subset R$ be an ideal and consider its radical

$$\sqrt{\mathfrak{a}} := \{a \in R \mid a^e \in \mathfrak{a}, e > 0\}$$

of $\mathfrak{a}$. Since R is noetherian, $\sqrt{\mathfrak{a}}$ is finitely generated, say by $a_1, \dots, a_n$; choose $e_i > 0$ such that $a_i^{e_i} \in \mathfrak{a}$. If we write $e := e_1 + \cdots + e_n$, then $(\sqrt{\mathfrak{a}})^e$ is generated by e-fold products of the generators, i.e.

$$(\sqrt{\mathfrak{a}})^e = \langle a_1^{r_1} \cdots a_n^{r_n} \mid r_1 + \cdots + r_n = e \rangle .$$

Now take any generator $a_1^{r_1} \cdots a_n^{r_n}$ of $(\sqrt{\mathfrak{a}})^e$. By the choice of e, we necessarily have $r_j \geq e_j$ for some $j = 1, \dots, n$. Thus $a_j^{r_j} = a_j^{e_j} a_j^{r_j - e_j} \in \mathfrak{a}$ and hence $a_1^{r_1} \cdots a_n^{r_n} \in \mathfrak{a}$. This proves that $(\sqrt{\mathfrak{a}})^e \subseteq \mathfrak{a}$.

Next we compute the radical. Since $\sqrt{\mathfrak{a}}$ is equal to be the intersection of all prime ideals of R containing $\mathfrak{a}$, then $\sqrt{\mathfrak{a}} = \mathfrak{m}$. Indeed, if $\mathfrak{p} \subseteq \mathfrak{m}$ is a nonzero prime ideal of R, then $\dim R = 1$ implies that $\mathfrak{p} = \mathfrak{m}$. This claim together with the above tells us that

$$\exists\, m > 0 \text{ such that } \mathfrak{m}^m \subseteq \mathfrak{a} \subseteq \mathfrak{m}.$$

This implies that there exists an exponent $0 < r \leq m$ for which $\mathfrak{a} \subseteq \mathfrak{m}^r$ but $\mathfrak{a} \not\subseteq \mathfrak{m}^{r+1}$. Thus, if $a \in \mathfrak{a} \subseteq \mathfrak{m}^r = (x)^r = (x^r)$, there exists $b \in R$ such that $a = bx^r$. Since $a \notin \mathfrak{m}^{r+1} = (x^{r+1})$, then $b \notin (x) = \mathfrak{m}$ and is thus a unit since R is local. Thus $x^r = b^{-1}a \in \mathfrak{a}$ and hence $\mathfrak{a} = (x^r)$. We have shown that every ideal of R is principal and of the form $\mathfrak{a} = \mathfrak{m}^r$ for some $r > 0$.

$(ii \implies i)$ Since R is a PID then $\mathfrak{m} = (x)$ for some $x \in R$. If $\mathfrak{m}/\mathfrak{m}^2 = 0$ then $\mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}$ so that Nakayam's Lemma implies that $\mathfrak{m} = 0$. However, this implies that R is a field which contradicts the fact that $\dim R = 1$. Thus $\mathfrak{m}/\mathfrak{m}^2 \neq 0$ and in particular $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq 1$.

Next, let $\{x_1 + \mathfrak{m}^2, \dots, x_n + \mathfrak{m}^2\}$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's Lemma again, we have $\mathfrak{m} = (x_1, \dots, x_n)$ so that $x_i = r_i x^{e_i}$ for some $r_i \in R$ and $e_i \geq 1$. If $e_i > 1$, then $x_i \in \mathfrak{m}^{e_i} \subseteq \mathfrak{m}^2$ and thus we would have $x_i + \mathfrak{m}^2 = 0$ which contradicts the fact that it is an element of a basis. Thus $e_i = 1$ and $x_i = r_i x$ for all $i = 1, \dots, n$. Similarly we must have that $r_i \notin \mathfrak{m}$. This means that $r_i + \mathfrak{m} \in k$ is invertible and hence

$$\{(r_1 + \mathfrak{m})^{-1}(x_1 + \mathfrak{m}^2), \dots, (r_n + \mathfrak{m})^{-1}(x_n + \mathfrak{m}^2)\} = \{\underbrace{x + \mathfrak{m}^2, \dots, x + \mathfrak{m}^2}_{n \text{ times}}\}$$

is also a basis for $\mathfrak{m}/\mathfrak{m}^2$. This is only possible if $n = 1$. Therefore $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$ as required.

□

A.1.2 Canonical Isomorphisms

Lemma A.4. *(Exercise 10 of Chapter II of [AM94]) Let R be a ring and $I \subset R$ an ideal contained in the Jacobson radical of R. Let M be an R-module, N be a finitely generated R-module and $u : M \to N$ an R-module homomorphism. If the induced map $\overline{u} : M/IM \to N/IN$ is surjective, then u is surjective.*

Proof. Since N is finitely generated, then $N = \langle n_1, \dots, n_t \rangle$ for some $n_i \in N$. In particular N/IN is finitely generated by $n_i + IN$. Since $\overline{u}$ is surjective, for each $i = 1, \dots, t$ there exists $m_i \in M$ such that $u(m_i) + \mathfrak{m}M = n_i + IN$. Since I is contained in the Jacobson radical of R, then Nakayama's Lemma allows us to lift the generating set $\{u(m_1) + IN, \dots, u(m_t) + IN\}$ of N/IN to a generating set $\{u(m_1), \dots, u(m_t)\}$ of N. Therefore $N = \langle u(m_1), \dots, u(m_t) \rangle \subseteq \operatorname{im}(u)$ and thus u is surjective. □

Proposition A.5. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let M and N be finitely generated R-modules with N free and let $\varphi : M \to N$ be an R-linear map. Suppose that the induced map $\overline{\varphi} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism. Then φ itself is an isomorphism.*

Proof. By the previous lemma, we only need to show that φ is injective and to do this, we will construct a left inverse.

Suppose that N is freely generated by $\{n_1, \ldots, n_s\} \subset N$ and let $m_1, \ldots, m_s \in M$ such that

$$m_i + \mathfrak{m}M = \overline{\varphi}^{-1}(n_i + \mathfrak{m}N).$$

Notice that this is equivalent to $\varphi(m_i) + \mathfrak{m}N = n_i + \mathfrak{m}N$. Since N is free, we obtain an R-linear map

$$\varphi' : N \longrightarrow M \quad \text{defined by} \quad n_i \mapsto m_i.$$

We claim that $\overline{\varphi'} \circ \overline{\varphi} = \mathrm{id}_{M/\mathfrak{m}M}$. To show this, let $m \in M$ and write $\varphi(m)$ in terms of the basis of N as $\varphi(m) = \sum r_i n_i$. Notice that by construction we have:

$$\overline{\varphi}\left((m - \sum r_i m_i) + \mathfrak{m}M\right) = \left(\varphi(m) - \sum r_i \varphi(m_i)\right) + \mathfrak{m}N = \left(\varphi(m) - \sum r_i n_i\right) + \mathfrak{m}N = 0,$$

so that $m + \mathfrak{m}M = \sum r_i m_i + \mathfrak{m}M$ since $\overline{\varphi}$ is an isomorphism. Thus

$$\overline{\varphi'}(\overline{\varphi}(m + \mathfrak{m}M)) = \overline{\varphi'}\left(\sum r_i n_i + \mathfrak{m}N\right) = \sum r_i m_i + \mathfrak{m}M = m + \mathfrak{m}M,$$

as claimed.

Next, we define

$$\psi : M \longrightarrow M \quad \text{with} \quad \psi = \varphi' \circ \varphi.$$

The previous lemma applied to $u = \psi$ implies that ψ is surjective since $\overline{\psi}$ is surjective. This means that if we consider M as an $R[X]$-module via ψ, i.e. $X \cdot m = \psi(m)$, then $M = (X)M$ so that the Cayley-Hamilton theorem tells us that there is a polynomial $f(X) \in R[x]$ with $f(X) \equiv 1 \pmod{X}$ such that $f(X) \cdot M = 0$ (see [AM94, Corollary 2.5]). In particular, there is another polynomial $g(X) \in R[X]$ such that $f(X) = 1 - Xg(X)$, and thus for every $m \in M$ we have

$$g(\psi)\psi(m) = g(X)X \cdot m = (1 - f(X)) \cdot m = m - f(X) \cdot m = m,$$

that is $g(\psi)\psi = \mathrm{id}_M$, or equivalently $(g(\psi)\varphi') \circ \varphi = \mathrm{id}_M$. This shows that φ has a left inverse as required. □

Proposition A.6. *Let S be a commutative ring. Let A, B and C be S-modules with A finitely generated and projective. Then the map*

$$\alpha : \mathrm{Hom}_S(B, C) \otimes_S A \longrightarrow \mathrm{Hom}_S\big(\mathrm{Hom}_S(A, B), C\big) \quad \textit{defined by} \quad \alpha(f \otimes a)(g) = f(g(a))$$

is a natural S-module isomorphism.

Proposition A.7. *Let R be a commutative unital ring and R' a commutative unital R-algebra. Let M be an R-module. Then there is an R'-algebra homomorphism*

$$\beta : R' \otimes_R \mathrm{End}_{R\text{-mod}}(M) \longrightarrow \mathrm{End}_{R'\text{-mod}}(R' \otimes_R M),$$

that satisfies

$$\beta(1 \otimes \varphi) = id_{R'} \otimes \varphi \tag{A.3}$$

for all $\varphi \in \mathrm{End}_{R\text{-mod}}(M)$. In fact, β, viewed as an R'-module homomorphism, is unique among all R'-module homomorphisms $R' \otimes_R \mathrm{End}_{R\text{-mod}}(M) \to \mathrm{End}_{R'\text{-mod}}(R' \otimes_R M)$ that satisfy (A.3)*. Furthermore, if M is a free R-module of finite rank, then β is an isomorphism.*

Proof. First we define β. Let $r \otimes \varphi$ be a simple tensor in $R' \otimes_R \mathrm{End}_{R\text{-mod}}(M)$. Next we define the map

$$\overline{\psi_{r\otimes\varphi}} : R' \times M \longrightarrow R' \otimes_R M \quad \text{with} \quad (x, m) \mapsto rx \otimes \varphi(m)$$

It is straightforward to check that $\overline{\psi_{r\otimes\varphi}}$ is R-bilinear (since R is commutative and φ is R-linear) so the universal property of the tensor product gives us a (unique) R-linear map

$$\psi_{r\otimes\varphi} : R' \otimes_R M \longrightarrow R' \otimes_R M \quad \text{with} \quad x \otimes m \mapsto rx \otimes \varphi(m).$$

In fact, $\psi_{r\otimes\varphi}$ is an R'-module endomorphism of $R' \otimes_R M$. Indeed, if $r' \in R'$ and $x \otimes m \in R' \otimes_R M$ is a simple tensor, then

$$\psi_{r\otimes\varphi}\big(r'(x \otimes m)\big) = \psi_{r\otimes\varphi}(r'x \otimes m) = rr'x \otimes \varphi(m) = r'rx \otimes \varphi(m) = r'(rx \otimes \varphi(m)) = r'\psi_{r\otimes\varphi}(x, m).$$

Hence $\psi_{r\otimes\varphi} \in \mathrm{End}_{R\text{-mod}}(R' \otimes_R M)$ and we get a map,

$$\beta : R' \otimes_R \mathrm{End}_{R\text{-mod}}(M) \longrightarrow \mathrm{End}_{R'\text{-mod}}(R' \otimes_R M) \quad \text{defined by} \quad r \otimes \varphi \mapsto \psi_{r\otimes\varphi} \tag{A.4}$$

which we extend to an R'-linear map.

Notice that β is actually an R'-algebra homomorphism. First of all, β preserves the mulitplicative identities since

$$\beta(1 \otimes \mathrm{id}_M)(x \otimes m) = \psi_{1\otimes \mathrm{id}_M}(x \otimes m) = 1 \cdot x \otimes \mathrm{id}_M(m) = x \otimes m = \mathrm{id}_{R'\otimes M}(x \otimes m)$$

so $\beta(1 \otimes \mathrm{id}_M) = \mathrm{id}_{R'\otimes M}$. And second of all, β is multiplicative since

$$\beta\big((r \otimes \varphi) \cdot (r' \otimes \varphi')\big) = \beta(rr' \otimes \varphi\varphi') = \psi_{rr'\otimes\varphi\varphi'}$$

and if $x \otimes m \in R' \otimes_R M$ is an arbitrary simple tensor, then

$$\psi_{r\otimes\varphi}\big(\psi_{r'\otimes\varphi'}(x \otimes m)\big) = \psi_{r\otimes\varphi}\big(r'x \otimes \varphi'(m)\big) = rr'x \otimes \varphi(\varphi'(m)) = \psi_{rr'\otimes\varphi\varphi'}(x \otimes m),$$
$$\therefore \qquad \beta\big((r \otimes \varphi) \cdot (r' \otimes \varphi')\big) = \psi_{rr'\otimes\varphi\varphi'} = \psi_{r\otimes\varphi}\psi_{r'\otimes\varphi'} = \beta(r \otimes \varphi)\beta(r' \otimes \varphi').$$

Next we show that β is unique among R'-module homomorphisms $R' \otimes_R \mathrm{End}_{R\text{-mod}}(M) \longrightarrow \mathrm{End}_{R'\text{-mod}}(R' \otimes_R M)$ satisfying A.3. To see this, let β' be one such R'-module homomorphism. If $r \otimes \varphi$ is any simple tensor in the domain, then

$$\beta'(r \otimes \varphi) = \beta'\big(r(1 \otimes \varphi)\big) = r\beta'(1 \otimes \varphi) = r(\mathrm{id}_{R'} \otimes \varphi) = r\beta(1 \otimes \varphi) = \beta(r \otimes \varphi).$$

Hence $\beta' = \beta$ and we conclude that β is the unique R'-linear map that satisfies (A.3).

For the last part of the proof, we assume that M is a free R-module, say $M \cong R^N$ for some $N \in \mathbb{N}$. We will exhibit another R'-module homomorphism $R' \otimes_R \mathrm{End}_{R\text{-mod}}(M) \to \mathrm{End}_{R'\text{-mod}}(R' \otimes_R M)$ which we will know to be an isomorphism (since it will arise from the functorial properties of the Hom and tensor functors $\mathrm{Hom}_{R\text{-mod}}(-, M)$ and $- \otimes_R M$) and then show that it satisfies (A.3). By the uniqueness property we showed above, we will have that this new isomorphism is in fact equal to β and thus β is an isomorphism as required.

To this end, let $\{m_1, \dots, m_N\}$ be a free generating set of M as an R-module. With this choice of basis, we have an R-linear isomorphism

$$\Phi : \mathrm{End}_R(M) \longrightarrow \mathrm{Mat}_{N\times N}(R)$$

which associates an endomorphism $\varphi : M \to M$ with its associated matrix. On the other hand, $R' \otimes_R M$ is isomorphic to $R' \otimes_R R^N \cong (R' \otimes_R R)^N \cong (R')^N$ via the isomorphism $M \cong R^N$ given by our choice of basis above. More precisely, given an arbitrary $m \in M$, we may write $m = \sum r_i m_i$ uniquely, so the composition

$$\begin{array}{ccccccc} R' \otimes_R M & \xrightarrow{\sim} & R' \otimes_R R^N & \xrightarrow{\sim} & (R' \otimes_R R)^N & \xrightarrow{\sim} & (R')^N \\ r' \otimes \left(\sum r_i m_i\right) & \longmapsto & r' \otimes (r_1, \dots, r_N) & \longmapsto & (r' \otimes r_1, \dots, r' \otimes r_N) & \longmapsto & (r'r_1, \dots, r'r_N) \end{array}$$

is an R-module isomorphism which is clearly also R'-linear. Hence, the canonical R'-basis $\{e_1, \dots, e_N\}$ of $(R')^N$ (where $e_1 = (1, 0, \dots, 0)$, etc.) pulls back to the R'-basis $\{1 \otimes m_1, \dots, 1 \otimes m_N\}$ of $R' \otimes_R M$. Thus as above, we obtain an R'-module isomorphism

$$\Psi : \mathrm{End}_{R'}(R' \otimes_R M) \longrightarrow \mathrm{Mat}_{N\times N}(R')$$

that associates an endomorphism $\psi : R' \otimes_R M \to R' \otimes_R M$ to its associated matrix with respect to the basis $\{1 \otimes m_1, \dots, 1 \otimes m_N\}$ of $R' \otimes_R M$.

Now we can put these two isomorphisms together. First we tensor Φ with R'. Since tensoring on the left by R' over R is a functor, then we get the R'-module isomorphism

$$\mathrm{id}_{R'} \otimes \Phi : R' \otimes_R \mathrm{End}_R(M) \xrightarrow{\sim} R' \otimes_R \mathrm{Mat}_{N\times N}(R).$$

Next, we know that $\mathrm{Mat}_{N\times N}(R) \cong R^{N^2}$ so we have canonical R'-module isomorphisms

$$\begin{array}{ccccccccc} R' \otimes_R \mathrm{Mat}_{N\times N}(R) & \xrightarrow{\sim} & R' \otimes_R R^{N^2} & \xrightarrow{\sim} & (R' \otimes_R R)^{N^2} & \xrightarrow{\sim} & (R')^{N^2} & \xrightarrow{\sim} & \mathrm{Mat}_{N\times N}(R') \\ r' \otimes (r_{ij}) & \longmapsto & r' \otimes (r_{11}, \dots, r_{NN}) & \mapsto & (r' \otimes r_{11}, \dots, r' \otimes r_{NN}) & \mapsto & (r'r_{11}, \dots, r'r_{NN}) & \longmapsto & (r'r_{ij}) \end{array}$$

If we call the composition of these canonical isomorphisms

$$\Xi : R' \otimes_R \mathrm{Mat}_{N\times N}(R) \xrightarrow{\sim} \mathrm{Mat}_{N\times N}(R') \quad \text{defined by} \quad r' \otimes A \mapsto r'A,$$

Then putting everything together gives us an R'-module isomorphism

$$\begin{array}{ccccccc} R'\otimes_R \mathrm{End}_R(M) & \xrightarrow{\mathrm{id}_{R'}\otimes\Phi} & R'\otimes_R \mathrm{Mat}_{N\times N}(R) & \xrightarrow{\Xi} & \mathrm{Mat}_{N\times N}(R') & \xrightarrow{\Psi^{-1}} & \mathrm{End}_{R'}(R'\otimes_R M) \\ r'\otimes\varphi & \longmapsto & r'\otimes\Phi(\varphi) & \longmapsto & r'\Phi(\varphi) & \longmapsto & \psi \end{array}$$

where the endomorphism $\psi : R'\otimes_R M \to R'\otimes_R M$ is defined by the matrix $r'\Phi(\varphi)$. Finally, under this map, $1\otimes\varphi$ gets mapped to multiplication by the matrix $1\cdot\Phi(\varphi) = \Phi(\varphi)$ which is the same matrix as the one associated to φ just viewed as a matrix with entries in R' (instead of R). Thus $1\otimes\varphi \mapsto \mathrm{id}_{R'}\otimes\varphi$ as required. This finishes the proof. □

Remark. This result is a particular case of a more general result that states that if M and N are any R-modules, there is a unique R'-module homomorphism

$$R'\otimes \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{R'}(R'\otimes_R M, R'\otimes_R N),$$

that satisfies $1\otimes\varphi \mapsto \mathrm{id}_{R'}\otimes\varphi$. Furthermore, if R' is a flat R-module and M is finitely presented, then this map is an isomorphism (see Proposition 2.10 of [Eis95]).

Corollary A.8. *Let M be a finitely generated R-module where R is a neotherian integral domain with fraction field K. Then there is a canonical K-algebra isomorphism:*

$$K\otimes_R \mathrm{End}_R(M) \xrightarrow{\sim} \mathrm{End}_K(K\otimes_R M)$$

Proof. Since K is the localization of R, it is a flat R-algebra. Since M is finitely generated, it admits an R-module epimorphism $R^N \twoheadrightarrow M$ for some $N\in\mathbb{N}$. Since R^N is a finitely generated module over a *noetherian* ring, then it is itself a noetherian R-module and hence the kernel of $R^N \twoheadrightarrow M$ is finitely generated. Thus M is finitely presented and we may apply Proposition A.7 to $M = N$ and $R' = K$. □

Proposition A.9. *Let R be a commutative ring and M an R-module. The following are equivalent:*

- (*i*) *M is locally free of finite rank,*
- (*ii*) *M is finitely generated and projective,*
- (*iii*) *M is finitely presented and $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p}\subset R$,*
- (*iv*) *M is finitely presented and $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}\subset R$.*

Remark A.10. If R is noetherian, then M is finitely generated if and only if M is finitely presented. Indeed, M is finitely generated if, by definition, there is a surjective R-module homomorphism $R^n \twoheadrightarrow M$ for some $n>0$. Since R^n is finitely generated and R is noetherian, it is a noetherian R-module and hence the kernel N of $R^n \twoheadrightarrow M$ is finitely generated so it admits a surjective map R-module homomorphism $R^m \twoheadrightarrow N$ for some $m>0$. Therefore we obtain an exact sequence $R^m \to R^n \to M \to 0$, i.e. M is finitely presented.

Proof. (of Proposition A.9)

($i \Longrightarrow ii$) Assume that M is locally free, i.e. there exists an open cover of basic open sets of Spec$R$, say Spec$R = D(f_1)\cup\cdots\cup D(f_n)$ such that each localization $M_{f_i} = M\otimes_R R_{f_i}$ is a free R_{f_i}-module of rank $N_i>0$. Notice that we may take the cover to be finite since Spec$R$ is (quasi) compact.

Next, take $N := \max\{N_1,\ldots,N_n\}$ to be the maximum of these ranks. Then for each $i=1,\ldots,n$,

$$F_i := M_{f_i}\oplus R_{f_i}^{N-N_i}$$

is a free R_{f_i}-module of rank N. Hence

$$F := \bigoplus_{i=1}^n F_i \quad \text{is a free} \quad S := \bigoplus_{i=1}^n R_{f_i}\text{-module.}$$

Thus

$$M\otimes_R S \cong \bigoplus_{i=1}^n (M\otimes_R R_{f_i}) \cong \bigoplus_{i=1}^n M_{f_i} \hookrightarrow \bigoplus_{i=1}^n (M_{f_i}\oplus R_{f_i}^{N-N_i}) = F$$

implies that $M\otimes_R S$ is a direct summand of F, a free S-module of finite rank, and thus is a finitely generated and projective S-module. The key property of S that we will leverage to take these desired properties to M, instead of $M\otimes_R S$, is that S is a faithfully flat R-algebra. Indeed, localizations are exact functors so R_{f_i} is a faithfully flat R-algebra and since tensoring over R commutes with direct sums, then S is also faithfully flat.

We first turn out attention to the finite generation of M. Since $M \otimes_R S$ is finitely generated we may choose a generating set $\{x_1, \ldots, x_t\}$. Each generator x_i can be written as

$$x_i = \sum_{j=1}^{t_i} m_{ij} \otimes s_{ij} = \sum_{j=1}^{t_i} s_{ij}(m_{ij} \otimes 1)$$

so $\{m_{ij} \otimes 1 \mid i = 1, \ldots, t,\ j = 1, \ldots, t_i\}$ is also a finite generating set of $M \otimes_R S$. Thus without loss of generality, we may assume that each x_i is already of the form $m_i \otimes 1$. This gives us

$$R^t \longrightarrow M \quad \text{defined on the canonical basis as} \quad e_i \mapsto m_i.$$

Tensoring this map with S yields

$$R^t \otimes_R S \cong (R \otimes_R S)^t \cong S^t \longrightarrow M \otimes_R S$$

which is surjective since $\{m_1 \otimes 1, \ldots, m_t \otimes 1\}$ generates $M \otimes_R S$. That is, we have:

$$\begin{array}{ccccc}
R^t & \longrightarrow & M & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle -\otimes_R S} & & \big\downarrow{\scriptstyle -\otimes_R S} & & \\
S^t & \longrightarrow & M \otimes_R S & \longrightarrow & 0
\end{array}$$

Since the bottom row is exact and S is faithfully flat, then the top row is exact and we conclude that M is finitely generated.

Now we turn to the projectiveness of M, i.e. we need to show that the functor $\operatorname{Hom}_R(M, -)$ is right exact (we already know that the hom functor is left exact), i.e. given any exact sequence

$$A \xrightarrow{\alpha} B \longrightarrow 0$$

then

$$\operatorname{Hom}_R(M, A) \xrightarrow{\alpha_*} \operatorname{Hom}_R(M, B) \longrightarrow 0$$

is exact. Since S is faithfully flat, the exactness of the above sequence is equivalent to the exactness of

$$S \otimes_R \operatorname{Hom}_R(M, A) \xrightarrow{\mathrm{id}_S \otimes \alpha_*} S \otimes_R \operatorname{Hom}_R(M, B) \longrightarrow 0.$$

We can obtain this sequence from $A \to B \to 0$ by applying, in order, the functors functors $- \otimes_R S$ and $\operatorname{Hom}_S(M \otimes_R S, -)$ followed by the natural isomorphism of Proposition A.7:

$$\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \\
A \otimes_R S & \longrightarrow & B \otimes_R S & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \\
\operatorname{Hom}_S(M \otimes_R S, A \otimes_R S) & \longrightarrow & \operatorname{Hom}_S(M \otimes_R S, B \otimes_R S) & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle \wr} & & \big\downarrow{\scriptstyle \wr} & & \\
S \otimes_R \operatorname{Hom}_R(M, A) & \longrightarrow & S \otimes_R \operatorname{Hom}_R(M, B) & \longrightarrow & 0.
\end{array}$$

Since the top row is exact and both $- \otimes_R S$ and $\operatorname{Hom}_S(M \otimes_R S, -)$ are right exact functors, the latter being right exact because $M \otimes_R S$ is projective, the bottom row is also exact as required.

$(ii \implies iii)$ Lets suppose that M is finitely generated and projective. The first assumption tells us that there is an epimorphism $R^n \to M$ for some $n > 0$. So if we write $K = \ker \nu$, we have the exact sequence

$$0 \longrightarrow K \xrightarrow{\iota} R^n \longrightarrow M \longrightarrow 0 \tag{A.5}$$

Since M is projective, this exact sequence is right-split by applying the definition of projective to the diagram:

$$\begin{array}{ccccc}
 & \swarrow & M & & \\
 & & \big\downarrow{\scriptstyle \mathrm{id}_M} & & \\
R^n & \xrightarrow[\nu]{} & M & \longrightarrow & 0
\end{array}$$

Thus (A.5) is *left*-split, i.e. there is an R-module homomorphism $\sigma : R^n \to K$ such that $\sigma\iota = \mathrm{id}_K$, i.e. σ has a right inverse and is thus surjective. This means that K admits an epimorphism $R^n \twoheadrightarrow K$ is is thus finitely generated. This proves that M is finitely presented.

($iii \Longrightarrow iv$) This is trivial.

($iv \Longrightarrow i$) For each maximal ideal $\mathfrak{m} \subset R$, we have that $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module. Since M is finitely presented, we may localize its finite presentation to get

$$\begin{array}{ccccccc} R^m & \longrightarrow & R^n & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle -\otimes_R R_\mathfrak{m}} & & \downarrow{\scriptstyle -\otimes_R R_\mathfrak{m}} & & \downarrow{\scriptstyle -\otimes_R R_\mathfrak{m}} & & \\ R^m \otimes_R R_\mathfrak{m} & \longrightarrow & R^n \otimes_R R_\mathfrak{m} & \longrightarrow & M \otimes_R R_\mathfrak{m} & \longrightarrow & 0 \\ \downarrow{\wr} & & \downarrow{\wr} & & \downarrow{\wr} & & \\ R^m_\mathfrak{m} & \longrightarrow & R^n_\mathfrak{m} & \longrightarrow & M_\mathfrak{m} & \longrightarrow & 0. \end{array} \tag{A.6}$$

The bottom row is exact since localization is an exact functor and the isomorphisms between the second and third row are canonical. Hence $M_\mathfrak{m}$ is finitely presented and is thus a free $R_\mathfrak{m}$-module of *finite* rank.

Let $\{m_1/s_1, \dots, m_k/s_k\}$ be a free generating set with $m_i \in M$ and $s_i \in R \setminus \mathfrak{m}$. Notice that $\{m_1/1, \dots, m_k/1\}$ is also a free generating set. Indeed, if $m/s \in M$, then there exist $u_1/t_1, \dots, u_k/t_k \in M_\mathfrak{m}$ such that

$$\frac{m}{s} = \sum_{i=1}^{k} \frac{u_i}{t_i}\frac{m_i}{s_i} = \sum_{i=1}^{k} \frac{u_i}{t_i s_i}\frac{m_i}{1}$$

so that $\{m_1/1, \dots, m_k/1\}$ generates and it is linearly independt since

$$0 = \sum_{i=1}^{k} \frac{u_i}{t_i}\frac{m_i}{1} = \sum_{i=1}^{k} \frac{u_i s_i}{t_i}\frac{m_i}{s_i} \quad\Longrightarrow\quad 0 = \frac{u_i s_i}{t_i} \quad\Longrightarrow\quad \frac{u_i}{t_i}$$

because $a_i \in R_\mathfrak{m}^\times$.

Next, with the free generating set $\{m_1/1, \dots, m_k/1\}$ we have the R-module homomorphism

$$\varphi : R^k \longrightarrow M \quad \text{defined on the canonical basis by} \quad e_i \mapsto m_i.$$

If we write $K := \ker\varphi$ and $N := M/\varphi(R^k)$ for the kernel and cokernel of φ respectively, we have the exact sequence

$$0 \longrightarrow K \hookrightarrow R^k \xrightarrow{\varphi} M \twoheadrightarrow N \longrightarrow 0. \tag{A.7}$$

Similarly as in (A.6), we obtain the exact sequence

$$0 \longrightarrow K_\mathfrak{m} \hookrightarrow R^k_\mathfrak{m} \xrightarrow{\varphi_\mathfrak{m}} M_\mathfrak{m} \twoheadrightarrow N_\mathfrak{m} \longrightarrow 0. \tag{A.8}$$

where the localized map $\varphi_\mathfrak{m} = \varphi \otimes \mathrm{id}_{R_\mathfrak{m}}$ is defined as on the canonical basis by

$$\varphi_\mathfrak{m}(e_i/1) = (\varphi \otimes \mathrm{id}_{R_\mathfrak{m}})(e_i \otimes 1) = m_i \otimes 1$$

and hence maps the canonical basis onto our free generating set for $M_\mathfrak{m}$. Thus $\varphi_\mathfrak{m}$ is an isomorphism and in particular, the terms surrounding it in the exact sequence (A.6) are zero, i.e. $K_\mathfrak{m} = N_\mathfrak{m} = 0$.

Since N is a homomorphic image of a finitely generated (in fact finitely presented) R-module, it is itself finitely generated; let $\{n_1, \dots, n_l\}$ be a generating set. Since $N_\mathfrak{m} = 0$, then $n_1/1 = 0$ in $N_\mathfrak{m}$, that is, there exists an $g_i \in R \setminus \mathfrak{m}$ such that $g_i n_i = 0$. If we set $g := g_1 \cdots g_l$, then $g \in R \setminus \mathfrak{m}$ since $\mathfrak{m}$ is prime and $gn_i = 0$ for all $i = 1, \dots, l$. This implies that $gN = 0$ since it annihilates its generators. For the same reasons, the localized module N_g is zero.

We would also like to find some $h \in R \setminus \mathfrak{m}$ for which $K_h = 0$, however, we cannot immediately use the same argument we used for N since it is not obvious that K is finitely generated because the definition of being finitely presented only states that there *exists* a presentation whose kernel of relations is finitely generated, not that *every* presentation has finitely generated kernel. However, this *is* true; For the moment, we assume that K is finitely generated and we postpone the proof of this fact until the end.

Assuming that K is finitely generated, the same argument we used for N yields an $h \in R \setminus \mathfrak{m}$ such that $K_h = 0$. By Proposition A.1 (or more precisely, Remark A.2) we have that

$$K_f \cong K \otimes_R R_f \cong K \otimes_R (R_h \otimes_{R_h} (R_h)_{g/1}) \cong (K \otimes_R R_h) \otimes_{R_h} (R_h)_{g/1} \cong K_h \otimes_{R_g} R_f = 0.$$

Swapping K with N and h with g in the above computation gives $N_f = 0$. Thus if we localize the exact sequence (A.7) with respect to $\{f^e \mid n \in \mathbb{Z}\}$ we obtain the exact sequence

$$0 \longrightarrow \underbrace{K_f}_{=0} \hookrightarrow R_f^k \xrightarrow{\varphi_f} M_f \twoheadrightarrow \underbrace{N_f}_{=0} \longrightarrow 0.$$

and thus $M_f \cong R_f^k$.

In conclusion, we have found, for every maximal ideal $\mathfrak{m} \subset R$, an element $f \in R \setminus \mathfrak{m}$ such that M_f is a free R_f-module. Thus, for every point $\mathfrak{p}$ in Spec$R$, we can find a basic open neighborhood $D(f_\mathfrak{p})$ that contains (a point $\mathfrak{m}$ contained in the closure of the point) $\mathfrak{p}$. Since Spec$R$ is (quasi) compact, we may take finitely many $f_1, \ldots, f_l \in R$ such that Spec$R = D(f_1) \cup \cdots \cup D(f_l)$ and such that M_{f_i} is a free R_{f_i}-module. That is M is locally free as required.

The only thing left to show is that K is finitely generated. To do this we localize the exact sequence in (A.7) with respect to the multiplicative subset $\{g^e \mid e \in \mathbb{Z}\}$, to obtain the *short* exact sequence

$$0 \longrightarrow K_g \hookrightarrow R_g^k \xrightarrow{\varphi_g} M_g \longrightarrow N_g = 0.$$

The same argument we used to show that $M_\mathfrak{m}$ is finitely presented applies for M_g. In fact, we get an analogous exact sequence to the bottom row of (A.6) which we attach the one above as follows:

$$\begin{array}{ccccccccc} & & R_g^m & \xrightarrow{\xi} & R_g^n & \xrightarrow{\psi} & M_g & \longrightarrow & 0 \\ & & & & & & \Vert & & \\ 0 & \longrightarrow & K_g & \xrightarrow{\iota} & R_g^k & \xrightarrow{\varphi_g} & M_g & \longrightarrow & 0, \end{array} \tag{A.9}$$

where we have now labeled the previously unamed arrows.

Since R_g^n is a free R_g-module, it is projective and hence by definition the map $\psi : R_g^n \to M$ factors through φ_g, i.e. there is a map $\eta : R_g^n \to R_g^k$ that makes the following diagram commute:

$$\begin{array}{ccccc} & & R_g^n & & \\ & \overset{\eta}{\swarrow} & \downarrow \psi & & \\ R_g^k & \xrightarrow[\varphi_g]{} & M_g & \longrightarrow & 0 \end{array}$$

Now, observe that $\varphi_g \eta \xi = \psi \xi = 0$ since the top row of (A.9) is exact. This shows that the image of $\eta\xi$ is contained in $\ker \varphi_g = \operatorname{im}(\iota)$. Thus, since R_g^m is free and hence projective, the map $\eta\xi$ factors through the surjective map $K_g \to \iota(K_g)$, i.e. there is a map $\rho : R_g^m \to K_g$ such that makes the following diagram commute:

$$\begin{array}{ccccc} & & R_g^m & & \\ & \overset{\rho}{\swarrow} & \downarrow \eta\xi & & \\ K_g & \xrightarrow[\iota]{} & \iota(K_g) & \longrightarrow & 0 \end{array}$$

Thus we can fill out the diagram in (A.9) with the the maps η and ρ to get the diagram

$$\begin{array}{ccccccccc} & & R_g^m & \xrightarrow{\xi} & R_g^n & \xrightarrow{\psi} & M_g & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow \eta & & \downarrow \mathrm{id}_M & & \\ 0 & \longrightarrow & K_g & \xrightarrow{\iota} & R_g^k & \xrightarrow{\varphi_g} & M_g & & \end{array}$$

which commutes by the constructions of η and ρ. We can now apply the Snake Lemma (see for example [?]) which states that there is a long exact sequence

$$\ker \rho \longrightarrow \ker \eta \longrightarrow \ker \mathrm{id}_M = 0 \longrightarrow \mathrm{coker}\rho \longrightarrow \mathrm{coker}\eta \longrightarrow \mathrm{cokerid} = 0 \longrightarrow 0$$

and in particular coker$\rho \cong$ cokerη. Since cokerη is the homomorphic image of a finitely generated module, namely R_g^k, it is finitely generated, and hence cokerρ is finitely generated as well. Furthermore, imρ is also finitely generated since it is the homomorphic image of R_g^m. Thus we can sandwhich K_g into a short exact sequence

$$0 \longrightarrow \mathrm{im}(\rho) \longrightarrow K_g \longrightarrow \mathrm{coker}\rho \longrightarrow 0$$

where its neighbors are finitely generated. This implies that K_g is finitely generated. Indeed, choosing finite set in K_g whose image in cokerρ generates, together with a finite generating set of $\mathrm{im}(\rho) \subseteq K_g$ yields a finite generating set for K_g.

□

Proposition A.11. *Let R be a commutative ring, A and R-algebra and M a left $A \otimes_R A^{\mathrm{op}}$-algebra. Then there is a natural R-module isomorphism.*

$$\mathrm{Hom}_{A\otimes_R A^{\mathrm{op}}}(A, M) \xrightarrow{\sim} Z_M(A) = \{m \in M \mid a \cdot m = m \cdot a, \ \ \forall a \in A\} \quad \textit{defined by} \quad f \mapsto f(1_A).$$

A.2 Sheaves of $\mathcal{O}_X$-modules

Let $X = (X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be a coherent sheaf. If $x \in X$ a point, and $\mathcal{F}_x$ the stalk of $\mathcal{F}$ at x. Then

$$\mathcal{F}_x \text{ is naturally an } \mathcal{O}_{X,x}\text{-module.}$$

This module structure is defined as follows: if $[s] \in \mathcal{F}_x$ is represented by $s \in \mathcal{F}(U)$ and $[r] \in \mathcal{O}_{X,x}$ is represented by $r \in \mathcal{O}_X(V)$ for some other open neighborhoods U and V of x, then we define

$$[r] \cdot [s] := [r|_W s|_W] \quad \text{where} \quad W \subseteq U \cap V \text{ is a smaller open neighborhood of } x.$$

This is possible since $s|_W$ lies in $\mathcal{F}(W)$ which is itself an $\mathcal{O}_X(W)$-module and thus can be multiplied by $r|_W \in \mathcal{O}_X(W)$.

Proposition A.12. *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules and $x \in X$. Then:*

(i) $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$,

(ii) If $\mathcal{F}$ is coherent, then $\left(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right)_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$.

Proof. Since the stalks of presheaves and their sheafifications are the same, then we may work with the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

(i) By definition of the stalk of a presheaf, we have

$$\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x = \left(\lim_{U \ni x} \mathcal{F}(U)\right) \otimes_{\lim_{U\ni X} \mathcal{O}_X(U)} \left(\lim_{U \ni x} \mathcal{G}(U)\right) \cong \lim_{U \ni x} \left(\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)\right) \cong \left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}\right)_x,$$

where the first isomorphism is natural and follows from the universal properties of the direct limit and the tensor product (see Lemma 11.4 of [Cut18]).

(ii) For simplicity, we write $\mathcal{H} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Since $\mathcal{F}$ is coherent, there exists an open neighborhood U of x such that $\mathcal{F}|_U \cong \widetilde{M}$ where M is a finitely generated $\mathcal{O}_X(U)$-module. Since stalks don't change when restricting to U, then the statement follows from the natural isomorphism

$$\mathrm{Hom}_{S^{-1}\mathcal{O}_X(U)}(S^{-1}\mathcal{F}(U), S^{-1}\mathcal{G}(U)) \cong S^{-1}\big(\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))\big)$$

where $S \subset \mathcal{O}_X(U)$ is any multiplicatively closed set (see Proposition 2.10 of [Eis95]).

□

Corollary A.13. *Let $(f, f^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces and $\mathcal{A}$ an $\mathcal{O}_X$-module. Then by definition, for any $y \in Y$ there is a local homormophism $f_y^{\#} : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ which makes the stalk $\mathcal{A}_{f(y)}$ into an $\mathcal{O}_{X,f(y)}$-module. Then*

$$(f^*\mathcal{A})_y \cong \mathcal{A}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y}.$$

Proposition A.14. *(Exercise II.5.1 of [Har77]) Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{A}$ a locally free $\mathcal{O}_X$-module of finite rank. If we denote the* dual *of $\mathcal{A}$ as*

$$\check{\mathcal{A}} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X).$$

Then the following are true:

(i) $\check{\check{\mathcal{A}}} \cong \mathcal{A}$,

(ii) *If* $\mathcal{F}$ *is an* $\mathcal{O}_X$*-module, then*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{F}) \cong \check{\mathcal{A}} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

(iii) *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are* $\mathcal{O}_X$*-modules, then*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$$

(iv) *(Projection Formula) If* $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ *is a morphism of ringed spaces and* $\mathcal{F}$ *is an* $\mathcal{O}_Y$*-module, then there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{A}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}.$$

Corollary A.15. *Let* $f : Y \to X$ *be a morphism of noetherian schemes and let* $\mathcal{A}$ *be a locally free* $\mathcal{O}_X$*-module. Then*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f^*\mathcal{A}, f^*\mathcal{A}) \cong f^*\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A})$$

Proof. We will show that $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f^*\mathcal{A}, f^*\mathcal{A})$ and $f^*\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A})$ represent the same functor and hence Yoneda's Lemma would imply that these are naturally isomorphic. Let $\mathcal{G}$ be any $\mathcal{O}_Y$-module and consider the following chain of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}\left(\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f^*\mathcal{A}, f^*\mathcal{A}), \mathcal{G}\right) &\cong \mathrm{Hom}\left(f^*\mathcal{A} \otimes_{\mathcal{O}_Y} \widecheck{f^*\mathcal{A}}, \mathcal{G}\right) && \text{by Proposition A.14.(ii)}\\
&\cong \mathrm{Hom}\left(f^*\mathcal{A}, \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\widecheck{f^*\mathcal{A}}, \mathcal{G})\right) && \text{by the Hom-Tensor adjunction}\\
&\cong \mathrm{Hom}\left(f^*\mathcal{A}, \check{\check{\left(f^*\mathcal{A}\right)}} \otimes_{\mathcal{O}_Y} \mathcal{G}\right) && \text{by Proposition A.14.(ii)}\\
&\cong \mathrm{Hom}\left(f^*\mathcal{A}, f^*\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{G}\right) && \text{by Proposition A.14.(i)}\\
&\cong \mathrm{Hom}\left(\mathcal{A}, f_*(f^*\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{G})\right) && \text{by the } f_*\text{-}f^* \text{ adjunction}\\
&\cong \mathrm{Hom}\left(\mathcal{A}, \mathcal{A} \otimes_{\mathcal{O}_X} f_*\mathcal{G}\right) && \text{by the Projection formula}\\
&\cong \mathrm{Hom}\left(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A}), f_*\mathcal{G}\right) && \text{by the Hom-Tensor adjunction}\\
&\cong \mathrm{Hom}\left(f^*\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A}), \mathcal{G}\right) && \text{by the } f_*\text{-}f^* \text{ adjunction}
\end{aligned}$$

This proves that $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(f^*\mathcal{A}, f^*\mathcal{A})$ and $f^*\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A})$ represent the same functor so they are isomorphic as required. □

Proposition A.16. *(Exercise II.5.7 of [Har77]) Let* X *be a neotherian scheme and* $\mathcal{F}$ *a coherent sheaf.*

(i) *Let* $x \in X$*. If the stalk* $\mathcal{F}_x$ *is a free* $\mathcal{O}_{X,x}$*-module. There exists an open neighborhood* U *of* x *for which* $\mathcal{F}|_U$ *is a free* $\mathcal{O}_X$*-module.*

(ii) $\mathcal{F}$ *is a locally free* $\mathcal{O}_X$*-module if and only if* $\mathcal{F}_x$ *is a free* $\mathcal{O}_{X,x}$*-module for all* $x \in X$*.*

Proof.

(i) Let $x \in X$. Since $\mathcal{F}$ is coherent, there exists an open affine neighborhood $V = \mathrm{Spec} A$ of x such that $\mathcal{F}|_V \cong \widetilde{M}$ where M is a finitely generated A-module; let $m_1, \ldots, m_n \in M$ generate M as an A-module. The point x corresponds to a prime ideal $\mathfrak{p} \subset A$ so $\mathcal{O}_{X,x} \cong A_\mathfrak{p}$ and $\mathcal{F}_x \cong M_\mathfrak{p}$ as $A_\mathfrak{p}$-modules.

Under these identifications, are assumption on $\mathcal{F}_x$ says that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module; let $\{z_1, \ldots, z_d\}$ be a free generating set of $M_\mathfrak{p}$. Each element can be written in the form

$$z_i = \frac{a_i}{s_i} z_i'$$

where $a_i \in A$, $s_i \in A \setminus \mathfrak{p}$ and $z_i' \in M$; set $s := s_1 \cdots s_d$. For each generator m_i of M as an A-module, consider their images $\ell(m_i) \in M_\mathfrak{p}$ under the canonical localization map $\ell : M \to M_\mathfrak{p}$. Then for each of these, we may write

$$\ell(m_k) = \sum_{i=1}^{d} \frac{a_{k,i}}{u_{k,i}} z_i \quad \text{with} \quad a_{k,i} \in A,\ u_{k,i} \in A \setminus \mathfrak{p}$$

If we define

$$u = \prod_{k=1}^{n} \prod_{i=1}^{d} u_{k,i}$$

then $u \notin \mathfrak{p}$ since $u_{k,i} \notin \mathfrak{p}$ for all k and i. Therefore $f := su \in A \setminus \mathfrak{p}$ and we get a canonical map $A_f \to A_\mathfrak{p}$.

This means we can define:

$$\psi : A_f^d \longrightarrow M_f \quad \text{with} \quad \left(\frac{b_1}{f^{j_1}}, \ldots, \frac{b_d}{f^{j_d}}\right) \mapsto \sum_{i=1}^{d} \frac{b_i}{f^{j_i}} z_i.$$

This map is clearly A_f-linear. It is also injective since $\{z_1, \ldots, z_d\}$ is a *free* generating set. Finally, to show surjectivity, let $(a/f^N)m \in M_f$ be arbitrary. Since $\{m_1, \ldots, m_n\}$ generates M over A, there exists $c_1, \ldots, c_n \in A$ such that $m = \sum c_k m_k$. Thus

$$\frac{a}{f^N} m = \frac{a}{f^N} \sum_{k=1}^{n} c_k \left(\sum_{i=1}^{d} \frac{a_{k,i}}{u_{k,i}} z_i\right) = \sum_{i=1}^{d} \left(\sum_{k=1}^{n} \frac{a c_k}{f^N u_{k,i}}\right) z_j.$$

Since, $u_{k,i} \mid f$ by construction, then $f = u_{k,i} u'_{k,i}$ for some $u'_{k,i} \in A$ so that:

$$\frac{a c_k}{f^N u_{k,i}} = \frac{a c_k u'_{k,i}}{f^{N+1}} \in A_f.$$

Hence,

$$\psi\left(\sum_{k=1}^{n} \frac{a c_k u'_{k,1}}{f^{N+1}}, \ldots, \sum_{k=1}^{n} \frac{a c_k u'_{k,d}}{f^{N+1}}\right) = \frac{a}{f^N} m,$$

and ψ is surjective.

Thus M_f is a free A_f-module. Going back to $\mathcal{F}$, we can conclude that $\mathcal{F}(D(f)) \cong M_f$ is a free $\mathcal{O}_X(D(f)) \cong A_f$ module where $D(f)$ is the basic open neighborhood of x defined by $f \in A$. Since already $\mathcal{F}|_V \cong \widetilde{M}$ and $D(F) \subseteq V$, then $\mathcal{F}|_{D(f)}$ is a free $\mathcal{O}_X$-module as required.

(*ii*) The "only if" part is simply the previous item. Lets suppose that $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module and let $x \in X$. By assumption, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. We will show that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module.

To see this, let $[s] \in \mathcal{F}_x$ be any element represented by some section $s \in \mathcal{F}(V)$ where V is an open neighborhood of x. Since $[s] = [s|_{U\cap V}]$, we may assume that $V \subseteq U$. By assumption, $\mathcal{F}(V)$ is a free $\mathcal{O}_X(V)$-module; let $s_1, \ldots, s_n \in \mathcal{F}(V)$ be a free generating set. Then we claim that $[s_1], \ldots, [s_n] \in \mathcal{F}_x$ is a free generating set for $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$.

Now, since $s \in \mathcal{F}(V)$, there exist $a_1, \ldots, a_n \in \mathcal{O}_X(V)$ such that $s = \sum a_i s_i$ and thus

$$[s] = \sum_{i=1}^{n} [a_i][s_i].$$

Now, if $[s] = \sum b_i [s_i]$ is any other linear combination, we would have

$$[0] = \sum_{i=1}^{n} [(a_i - b_i) s_i]$$

which means that there exists an open neighborhood W of x such that $W \subseteq V$ and

$$0|_W = \left.\left(\sum_{i=1}^{n} (a_i - b_i) s_i\right)\right|_W$$

□

Corollary A.17. *Let $\mathcal{F}$ be a coherent locally free $\mathcal{O}_X$-module. Then*

$$\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{F}) \cong \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{F} \otimes_{O_X} \mathcal{F}).$$

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